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ARTICLE

Exact Solutions and Finite Time Stability of Linear Conformable Fractional Systems with Pure Delay

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ABSTRACT

We study nonhomogeneous systems of linear conformable fractional differential equations with pure delay. By using new conformable delayed matrix functions and the method of variation, we obtain a representation of their solutions. As an application, we derive a finite time stability result using the representation of solutions and a norm estimation of the conformable delayed matrix functions. The obtained results are new, and they extend and improve some existing ones. Finally, an example is presented to illustrate the validity of our theoretical results.

KEYWORDS

Representation of solutions; conformable fractional derivative; conformable delayed matrix function; conformable fractional delay differential equations; finite time stability

1 Introduction

In recent years, particularly in 2014, Khalil et al. [1] introduced a new definition of the fractional derivative called the conformable fractional derivative that extends the classical limit definition of the derivative of a function. The conformable fractional derivative has main advantages compared with other previous definitions. It can, for example, be used to solve the differential equations and systems exactly and numerically easily and efficiently, it satisfies the product rule and quotient rule, it has results similar to known theorems in classical calculus, and applications for conformable differential equations in a variety of fields have been extensively studied, see [2–10] and the references therein. On the other hand, in 2003, Khusainov et al. [11] represented the solutions of linear delay differential equations by constructing a new concept of a delayed exponential matrix function. In 2008, Khusainov et al. [12] adopted this approach to represent the solutions of an oscillating system with pure delay by establishing a delayed matrix sine and a delayed matrix cosine. This pioneering research yielded plenty of novel results on the representation of solutions, which are applied in the stability analysis and control problems of time-delay systems; see for example [13–28] and the references therein. Thereafter, in 2021, Xiao et al. [29] obtained the exact solutions of linear conformable fractional delay differential equations of order $\alpha \in (0,1]$ by constructing a new conformable delayed exponential matrix function.

However, to the best of our knowledge, no study exists dealing with the representation and stability of solutions of conformable fractional delay differential systems of order $\alpha \in (1,2]$.

Motivated by these papers, we consider the explicit formula of solutions of linear conformable fractional differential equations with pure delay

$$\begin{aligned} &\left(\mathfrak{D}_0^\alpha y\right)(x) = -By(x-\tau) + f(x), \quad \text{for x} \geq 0, \tau > 0, \\ &\quad y(x) \equiv \psi(x), y'(x) \equiv \psi'(x) \quad \text{for} -\tau \leq x \leq 0, \end{aligned} \tag{1}$$

by constructing new conformable delayed matrix functions. Moreover, the representation of solutions of Eq. (1) is used to obtain a finite time stability result on $W = [0, L]$, $L > 0$, where $\mathfrak{D}_0^\alpha$ is called the conformable fractional derivative of order $\alpha \in (1,2]$ with lower index zero, $y(x) \in \mathbb{R}^n$, $\psi \in C^2([-\tau, 0], \mathbb{R}^n)$, $B \in \mathbb{R}^{n\times n}$ is a constant nonzero matrix and $f \in C([0,\infty), \mathbb{R}^n)$ is a given function.

The paper is organized as follows: In Section 2, we present some basic definitions concerning conformable fractional derivative and finite time stability, and construct new conformable delayed matrix functions and derive their properties for use when we discuss the representation of solutions and finite time stability. In Section 3, by using the new conformable delayed matrix functions, we give the explicit formula of solutions of Eq. (1). In Section 4, as an application, we derive a finite time stability result using the representation of solutions. Finally, we give an example to illustrate the main results.

2 Preliminaries

Throughout the paper, we denote the vector norm and matrix norm, respectively, as $\|y\| = \sum_{i=1}^n |y_i|$ and $\|B\| = \max_{1\leq j\leq n} \sum\limits_{i=1}^n \left|b_{ij}\right|$; y_i and b_{ij} are the elements of the vector y and the matrix B, respectively. Denote $C(W, \mathbb{R}^n)$ the Banach space of vector-value continuous function from $W \to \mathbb{R}^n$ endowed with the norm $\|y\|_C = \max_{x\in W} \|y(x)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We introduce a space $C^1(W, \mathbb{R}^n) = \{y \in C(W, \mathbb{R}^n) : y' \in C(W, \mathbb{R}^n)\}$. Furthermore, we see $\|\psi\|_C = \max_{\upsilon\in[-\tau,0]} \|\psi(\upsilon)\|$.

We recall some basic definitions of conformable fractional derivative, fractional exponential function, and finite time stability.

Definition 2.1. ([2, Definition 2.2]). Let $f : [a,\infty) \to \mathbb{R}^n$ be a differentiable function at x. Then the conformable fractional derivative for f of order $\alpha = (1,2]$ is given by

$$\mathfrak{D}_a^\alpha(f)(x) = \lim_{\varepsilon\to 0} \frac{f'\left(x + \varepsilon(x-a)^{2-\alpha}\right) - f'(x)}{\varepsilon}, \quad x > \text{a},$$

if the limit exists.

Remark 2.1. As a consequence of Definition 2.1, we can show that

$$\mathfrak{D}_a^\alpha(f)(x) = (x-a)^{2-\alpha} f''(x),$$

where $\alpha = (1,2]$, and f is 2-differentiable at $x > a$.

Definition 2.2. ([2]). We define the fractional exponential function as follows:

$$E_{\alpha}(\lambda, x-a) = \exp\left(\lambda.\frac{(x-a)^{\alpha}}{\alpha}\right) = \sum_{k=0}^{\infty}\frac{\lambda^{k}(x-a)^{\alpha k}}{\alpha^{k}k!}, \quad \alpha > 0, \lambda \in \mathbb{R}.$$

Definition 2.3. ([30]). The system in Eq. (1) is finite time stable with respect to $\{0, W, \tau, \delta, \beta\}$, $\delta < \beta$ if and only if $\eta < \delta$ implies $\|y(x)\| < \beta$ for all $x \in W$, where $\eta = \max\left\{\|\psi\|_{C}, \|\psi'\|_{C}, \|\psi''\|_{C}\right\}$ and δ, β are real positive numbers.

Next, we construct new conformable delayed matrix functions that are the fundamental solution matrices of Eq. (1).

Definition 2.4. The conformable delayed matrix functions $\mathcal{H}_{\tau,\alpha}(Bx^{\alpha})$ and $\mathcal{M}_{\tau,\alpha}(Bx^{\alpha})$ are defined as:

$$\mathcal{H}_{\tau,\alpha}(Bx^{\alpha}) := \begin{cases} \Theta, & -\infty < x < -\tau, \\ I, & -\tau \le x < 0, \\ I - B\dfrac{1}{\alpha(\alpha-1)}x^{\alpha}, & 0 \le x < \tau, \\ \vdots & \vdots \\ I - B\dfrac{1}{\alpha(\alpha-1)}x^{\alpha} + B^{2}\dfrac{1}{2!\,\alpha^{2}(\alpha-1)(2\alpha-1)}(x-\tau)^{2\alpha} \\ \quad + \cdots + (-1)^{m}B^{m}\dfrac{1}{m!\,\alpha^{m}\prod_{i=1}^{m}(i\alpha-1)}(x-(m-1)\tau)^{m\alpha}, \\ & (\mathrm{m}-1)\tau \le x < \mathrm{m}\tau, \end{cases} \tag{2}$$

$$\mathcal{M}_{\tau,\alpha}(Bx^{\alpha}) := \begin{cases} \Theta, & -\infty < x < -\tau, \\ I(x+\tau), & -\tau \le x < 0, \\ I(x+\tau) - B\dfrac{1}{\alpha(\alpha+1)}x^{\alpha+1} & 0 \le x < \tau, \\ \vdots & \vdots \\ I(x+\tau) - B\dfrac{1}{\alpha(\alpha+1)}x^{\alpha+1} + B^{2}\dfrac{1}{2!\,\alpha^{2}(\alpha+1)(2\alpha+1)}(x-\tau)^{2\alpha+1} \\ \quad + \cdots + (-1)^{m}B^{m}\dfrac{1}{m!\,\alpha^{m}\prod_{i=1}^{m}(i\alpha+1)}(x-(m-1)\tau)^{m\alpha+1}, \\ & (\mathrm{m}-1)\tau \le x < \mathrm{m}\tau, \end{cases} \tag{3}$$

respectively, where $m = 0, 1, 2, \ldots$, I is the $n \times n$ identity matrix and Θ is the $n \times n$ null matrix.

Lemma 2.1. The following rule is true:

$$\mathfrak{D}_{0}^{\alpha}\mathcal{H}_{h,\alpha}(Bx^{\alpha}) = -B\mathcal{H}_{h,\alpha}\left(B(x-h)^{\alpha}\right).$$

Proof. First, when $x \in (-\infty, -\tau)$, we obtain $\mathcal{H}_{\tau,\alpha}(Bx^{\alpha}) = \mathcal{H}_{\tau,\alpha}(B(x-\tau)^{\alpha}) = \Theta$, and we can see that Lemma 2.1 holds. Following that, set $(m-1)\tau \le x < m\tau$, $m = 0, 1, 2, \ldots$, we get

$$\mathcal{H}_{\tau,\alpha}\left(Bx^{\alpha}\right) = I - B\frac{1}{\alpha\left(\alpha-1\right)}x^{\alpha} + B^{2}\frac{1}{2!\,\alpha^{2}\left(\alpha-1\right)\left(2\alpha-1\right)}\left(x-\tau\right)^{2\alpha} + \cdots + (-1)^{m}B^{m}\frac{1}{m!\,\alpha^{m}\prod_{i=1}^{m}\left(i\alpha-1\right)}\left(x-\left(m-1\right)\tau\right)^{m\alpha}.$$

Applying Remark 2.1, we get

$$\begin{aligned}
&\mathfrak{D}_{0}^{\alpha}\mathcal{H}_{\tau,\alpha}\left(Bx^{\alpha}\right)\\
&= \mathfrak{D}_{0}^{\alpha}I - \mathfrak{D}_{0}^{\alpha}\left[B\frac{1}{\alpha\left(\alpha-1\right)}x^{\alpha}\right] + \mathfrak{D}_{\tau}^{\alpha}\left[B^{2}\frac{1}{2!\,\alpha^{2}\left(\alpha-1\right)\left(2\alpha-1\right)}\left(x-\tau\right)^{2\alpha}\right]\\
&+ \cdots + \mathfrak{D}_{(m-1)\tau}^{\alpha}\left[(-1)^{m}B^{m}\frac{1}{m!\,\alpha^{m}\prod_{i=1}^{m}\left(i\alpha-1\right)}\left(x-\left(m-1\right)\tau\right)^{m\alpha}\right]\\
&= \Theta - B + B^{2}\frac{1}{\alpha\left(\alpha-1\right)}\left(x-\tau\right)^{\alpha} - B^{3}\frac{1}{2!\,\alpha^{2}\left(\alpha-1\right)\left(2\alpha-1\right)}\left(x-2\tau\right)^{2\alpha}\\
&+ \cdots + (-1)^{m}B^{m}\frac{1}{\left(m-1\right)!\,\alpha^{m-1}\prod_{i=1}^{m-1}\left(i\alpha-1\right)}\left(x-\left(m-1\right)\tau\right)^{(m-1)\alpha}\\
&= -B\left[I - B\frac{1}{\alpha\left(\alpha-1\right)}\left(x-\tau\right)^{\alpha} + B^{2}\frac{1}{2!\,\alpha^{2}\left(\alpha-1\right)\left(2\alpha-1\right)}\left(x-2\tau\right)^{2\alpha}\right.\\
&\left.+ \cdots + (-1)^{m-1}B^{m-1}\frac{1}{\left(m-1\right)!\,\alpha^{m-1}\prod_{i=1}^{m-1}\left(i\alpha-1\right)}\left(x-\left(m-1\right)\tau\right)^{(m-1)\alpha}\right]\\
&= -B\mathcal{H}_{\tau,\alpha}\left(B(x-\tau)^{\alpha}\right).
\end{aligned}$$

This completes the proof.

In the same way that we proved Lemma 2.1, we can derive the next result.

Lemma 2.2. The following rule is true:

$$\mathfrak{D}_{0}^{\alpha}\mathcal{M}_{h,\alpha}\left(Bx^{\alpha}\right) = -B\mathcal{M}_{h,\alpha}\left(B(x-h)^{\alpha}\right).$$

To conclude this section, we provide a norm estimation of the conformable delayed matrix functions, which is used while discussing finite time stability.

Lemma 2.3. For any $x \in \left[\left(m-1\right)\tau, m\tau\right]$, $m = 0, 1, 2, \ldots$, we have

$$\left\|\mathcal{H}_{\tau,\alpha}\left(Bx^{\alpha}\right)\right\| \leq E_{\alpha}\left(\frac{\|B\|}{\alpha-1}, x\right).$$

Proof. Taking the norm of Eq. (2), we get

$$\begin{aligned}\left\|\mathcal{H}_{\tau,\alpha}\left(Bx^{\alpha}\right)\right\| &\leq 1+\|B\|\frac{x^{\alpha}}{\alpha\left(\alpha-1\right)}+\|B\|^{2}\frac{\left(x-\tau\right)^{2\alpha}}{2!\,\alpha^{2}\left(\alpha-1\right)\left(2\alpha-1\right)}\\&+\cdots+\|B\|^{m}\frac{\left(x-\left(m-1\right)\tau\right)^{m\alpha}}{m!\,\alpha^{m}\prod\limits_{i=1}^{m}\left(i\alpha-1\right)}\\&\leq 1+\|B\|\frac{x^{\alpha}}{\alpha\left(\alpha-1\right)}+\|B\|^{2}\frac{x^{2\alpha}}{2!\,\alpha^{2}(\alpha-1)^{2}}\\&+\cdots+\|B\|^{m}\frac{x^{m\alpha}}{m!\,\alpha^{m}(\alpha-1)^{m}}\\&\leq\sum_{k=0}^{\infty}\frac{\|B\|^{k}x^{\alpha k}}{(\alpha-1)^{k}\alpha^{k}k!}=E_{\alpha}\left(\frac{\|B\|}{\alpha-1},x\right).\end{aligned}$$

This completes the proof.

Lemma 2.4. For any $x\in[(m-1)\,\tau,m\tau]$, $m=0,1,2,\ldots$, we have

$$\left\|\mathcal{M}_{\tau,\alpha}\left(Bx^{\alpha}\right)\right\|\leq\left(x+\tau\right)E_{\alpha}\left(\frac{\|B\|}{\alpha+1},x+\tau\right).$$

Proof. Taking the norm of Eq. (3), we get

$$\begin{aligned}\left\|\mathcal{M}_{\tau,\alpha}\left(Bx^{\alpha}\right)\right\| &\leq\left(x+\tau\right)+\|B\|\frac{1}{\alpha\left(\alpha+1\right)}x^{\alpha+1}\\&+\|B\|^{2}\frac{1}{2!\,\alpha^{2}\left(\alpha+1\right)\left(2\alpha+1\right)}\left(x-\tau\right)^{2\alpha+1}\\&+\cdots+\|B\|^{m}\frac{1}{m!\,\alpha^{m}\prod\limits_{i=1}^{m}\left(i\alpha+1\right)}\left(x-\left(m-1\right)\tau\right)^{m\alpha+1}\\&\leq\left(x+\tau\right)+\|B\|\frac{\left(x+\tau\right)^{\alpha+1}}{\alpha\left(\alpha+1\right)}+\|B\|^{2}\frac{\left(x+\tau\right)^{2\alpha+1}}{2\alpha^{2}(\alpha+1)^{2}}\\&+\cdots+\|B\|^{m}\frac{\left(x+\tau\right)^{m\alpha+1}}{m!\,\alpha^{m}(\alpha+1)^{m}}\\&\leq\sum_{k=0}^{\infty}\frac{\|B\|^{k}(x+\tau)^{k\alpha+1}}{k!\,\alpha^{k}(\alpha+1)^{k}}=\left(x+\tau\right)E_{\alpha}\left(\frac{\|B\|}{\alpha+1},x+\tau\right).\end{aligned}$$

This completes the proof.

3 Exact Solutions for Linear Conformable Fractional Delay Systems

In this section, we give the exact solutions of Eq. (1) via the conformable delayed matrix functions and the method of variation of constants. To do this, we consider the homogeneous system of linear conformable fractional delay differential equations

$$\begin{aligned}&\left(\mathfrak{D}_0^\alpha y\right)(x) = -By(x-\tau), \quad \text{for} \quad x \geq 0, \tau > 0,\\&y(x) \equiv \psi(x), y'(x) \equiv \psi'(x) \quad \text{for} \quad -\tau \leq x \leq 0,\end{aligned} \tag{4}$$

and the linear inhomogeneous conformable fractional delay system

$$\begin{aligned}&\left(\mathfrak{D}_0^\alpha y\right)(x) = -By(x-\tau) + f(x), \quad \text{for} \quad x \geq 0, \tau > 0,\\&\quad y(x) \equiv \Theta, y'(x) \equiv \Theta \quad \text{for} \quad -\tau \leq x \leq 0.\end{aligned} \tag{5}$$

Theorem 3.1. The solution $y(x)$ of Eq. (4) has the representation

$$y(x) = \begin{cases} \psi(x), & -\tau \leq x \leq 0,\\ \mathcal{H}_{\tau,\alpha}(Bx^\alpha)\psi(-\tau) + \mathcal{M}_{\tau,\alpha}(Bx^\alpha)\psi'(-\tau) & \\ \quad + \int_{-\tau}^0 \mathcal{M}_{\tau,\alpha}(\mathrm{B}(x-\tau-\upsilon)^\alpha)\upsilon^{\alpha-2}\mathfrak{D}_0^\alpha\psi(\upsilon)\,\mathrm{d}\upsilon, & x \geq 0.\end{cases} \tag{6}$$

Proof. We seek for a solution of Eq. (4) in the form

$$y(x) = \mathcal{H}_{\tau,\alpha}(Bx^\alpha)c_1 + \mathcal{M}_{\tau,\alpha}(Bx^\alpha)c_2 + \int_{-\tau}^0 \mathcal{M}_{\tau,\alpha}(B(x-\tau-\upsilon)^\alpha)\upsilon^{\alpha-2}\mathfrak{D}_0^\alpha r(\upsilon)\,d\upsilon, \tag{7}$$

or

$$y(x) = \mathcal{H}_{\tau,\alpha}(Bx^\alpha)c_1 + \mathcal{M}_{\tau,\alpha}(Bx^\alpha)c_2 + \int_{-\tau}^0 \mathcal{M}_{\tau,\alpha}(B(x-\tau-\upsilon)^\alpha)r''(\upsilon)\,d\upsilon,$$

where c_1 and c_2 are unknown constants vectors on $\mathbb{R}^n$, and $r(x)$ is an unkown twice continuously differentible vector function. From Lemmas 2.1 and 2.2, we deduce that $\mathcal{H}_{\tau,\alpha}(Bx^\alpha)$ and $\mathcal{M}_{\tau,\alpha}(Bx^\alpha)$ are solutions of Eq. (4). We notice that Eq. (6) is a solution of Eq. (4) due to the linearity of solutions for arbitrary c_1, c_2 and $r(x)$. Now we find the constants c_1 and c_2, and the vector function $r(x)$ so that the initial conditions $y(x) \equiv \psi(x)$, $y(x) \equiv \psi'(x)$ for $-\tau \leq x \leq 0$, are satisfied. That is, the following relations hold for $-\tau \leq x \leq 0$:

$$\mathcal{H}_{\tau,\alpha}(Bx^\alpha)c_1 + \mathcal{M}_{\tau,\alpha}(Bx^\alpha)c_2 + \int_{-\tau}^0 \mathcal{M}_{\tau,\alpha}(B(x-\tau-\upsilon)^\alpha)r''(\upsilon)\,d\upsilon = \psi(x), \tag{8}$$

and

$$\frac{d}{dx}\left\{\mathcal{H}_{\tau,\alpha}(Bx^\alpha)c_1 + \mathcal{M}_{\tau,\alpha}(Bx^\alpha)c_2 + \int_{-\tau}^0 \mathcal{M}_{\tau,\alpha}(B(x-\tau-\upsilon)^\alpha)r''(\upsilon)\,d\upsilon\right\} = \psi'(x). \tag{9}$$

Consider Eq. (8). If $-\tau \leq x < 0$, then

$$\mathcal{H}_{\tau,\alpha}(Bx^\alpha) = I, \quad \mathcal{M}_{\tau,\alpha}(\mathrm{B}x^\alpha) = \mathrm{I}(\mathrm{x}+\tau),$$

and

$$\mathcal{M}_{\tau,\alpha}(B(x-\tau-\upsilon)^\alpha) = \begin{cases} I(x-\upsilon), & \upsilon \in [-\tau, x],\\ \Theta, & \upsilon \in (x, 0],\end{cases}$$

which implies that

$$c_1 + (x+\tau)c_2 + \int_{-\tau}^x (x-\upsilon)r''(\upsilon)\,d\upsilon = \psi(x), \tag{10}$$

and

$$\int_{-\tau}^x (x-\upsilon)r''(\upsilon)\,d\upsilon = -(x+\tau)r'(-\tau) + r(x) - r(-\tau). \tag{11}$$

Substituting Eq. (11) into Eq. (10), we get

$$(c_1 - r(-\tau)) + (c_2 - r'(-\tau))(x+\tau) + (r(x) - \psi(x)) = \Theta. \tag{12}$$

Differentiating Eq. (12) with respect to x, we have

$$(c_2 - r'(-\tau)) + (r'(x) - \psi'(x)) = \Theta. \tag{13}$$

As a result, we find that the equalities obtained Eqs. (12) and (13) are true if

$$c_1 = \psi(-\tau), \quad \mathrm{c}_2 = \psi'(-\tau), \quad \mathrm{r}(x) = \psi(x). \tag{14}$$

Substituting Eq. (14) into Eq. (7), we obtain Eq. (6). This finishes the proof.

Theorem 3.2. The particular solution $y_0(x)$ of Eq. (5) has the representation

$$y_0(x) = \int_0^x \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\upsilon^{\alpha-2} f(\upsilon)\,d\upsilon. \tag{15}$$

Proof. We try to find a particular solution $y_0(x)$ of Eq. (5) in the form

$$y_0(x) = \int_0^x \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\xi(\upsilon)\,d\upsilon, \tag{16}$$

by applying the method of variation of constants, where $\xi(\upsilon)$, $0 < \upsilon \le x$, is an unknown function. Taking the conformable derivative of Eq. (16), we get

$$\begin{aligned}\mathfrak{D}_0^\alpha y_0(x) &= \int_0^x \mathfrak{D}_0^\alpha \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-s)^\alpha\right)\xi(\upsilon)\,d\upsilon + x^{2-\alpha}\xi(x)\\ &= -B\int_0^x \mathcal{M}_{\tau,\alpha}\left(B(x-2\tau-\upsilon)^\alpha\right)\xi(\upsilon)\,d\upsilon + x^{2-\alpha}\xi(x).\end{aligned} \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (5), and noting that

$$\int_{x-\tau}^x \mathcal{M}_{\tau,\alpha}\left(B(x-2\tau-\upsilon)^\alpha\right)\xi(\upsilon)\,d\upsilon = \Theta,$$

We have $x^{2-\alpha}\xi(x) = f(x)$. Substituting $\xi(x) = x^{\alpha-2}f(x)$ into Eq. (16), we obtain Eq. (15). This completes the proof.

Corollary 3.1. The solution $y(x)$ of Eq. (1) can be represented as

$$y(x) = \begin{cases} \psi(x), & -\tau \le x \le 0,\\ \mathcal{H}_{\tau,\alpha}(Bx^\alpha)\psi(-\tau) + \mathcal{M}_{\tau,\alpha}(Bx^\alpha)\psi'(-\tau) & \\ +\int_{-\tau}^0 \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\upsilon^{\alpha-2}\mathfrak{D}_0^\alpha\psi(\upsilon)\,d\upsilon & \\ +\int_0^x \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\upsilon^{\alpha-2}f(\upsilon)\,d\upsilon, & x \ge 0. \end{cases} \tag{18}$$

Remark 3.1. Let $\alpha = 2$ in Eq. (1). Then Corollary 3.1 coincides with Corollary 1 in [13].

Remark 3.2. Let $\alpha = 2$, $B = B^2$ in Eq. (1) such that the matrix B is a nonsingular $n \times n$ matrix. Then

$$\mathcal{H}_{\tau,2}\left(B^2x^2\right) = \cos_\tau(Bx), \quad \mathcal{M}_{\tau,2}\left(\mathrm{B}^2x^2\right) = \mathrm{B}^{-1}\sin_\tau(\mathrm{Bx}).$$

where $\cos_\tau(Bx)$ and $\sin_\tau(Bx)$ are called the delayed matrix of cosine and sine type, respectively, defined in [12]. Therefore, Corollary 3.1 coincides with Theorems 1 and 2 in [12].

4 Finite Time Stability of Linear Conformable Fractional Delay Systems

In this section, we establish some sufficient conditions for the finite time stability results of Eq. (1) by using a norm estimation of the conformable delayed matrix functions and the formula of general solutions of Eq. (1).

Theorem 4.1. The system Eq. (1) is finite time stable with respect to $\{0, W, \tau, \delta, \beta\}$, $\delta < \beta$ if

$$E_\alpha\left(\frac{\|B\|}{\alpha+1}, L+\tau\right) < \frac{\beta - \delta E_\alpha\left(\frac{\|B\|}{\alpha-1}, L\right) - \frac{\|f\|_C}{\alpha(\alpha-1)} L^\alpha E_\alpha\left(\frac{\|B\|}{\alpha+1}, L\right)}{\delta\,(L+\tau)\,(\tau+1)}. \tag{19}$$

Proof. By using Definition 2.3, and Theorems 3.1 and 3.2, we have $\eta < \delta$ and

$$\begin{aligned}
\|y(x)\| \le{}& \left\|\mathcal{H}_{\tau,\alpha}(Bx^\alpha)\right\| \|\psi(-\tau)\| + \left\|\mathcal{M}_{\tau,\alpha}(Bx^\alpha)\right\| \|\psi'(-\tau)\| \\
&+ \left\|\int_{-\tau}^{0} \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\psi''(\upsilon)\,d\upsilon\right\| \\
&+ \left\|\int_{0}^{x} \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\upsilon^{\alpha-2} f(\upsilon)\,d\upsilon\right\| \\
\le{}& \delta\left\|\mathcal{H}_{\tau,\alpha}(Bx^\alpha)\right\| + \delta\left\|\mathcal{M}_{\tau,\alpha}(Bx^\alpha)\right\| \\
&+ \delta\int_{-\tau}^{0}\left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\right\| d\upsilon \\
&+ \|f\|_C \int_{0}^{x}\left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\right\| \upsilon^{\alpha-2} d\upsilon.
\end{aligned} \tag{20}$$

Note that $\mathcal{M}_{\tau,\alpha}(Bx^\alpha) = \Theta$ if $x \in (-\infty, -\tau)$. For $-\tau \le \upsilon \le 0$, we get

$$\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right) = \begin{cases} \mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right), & \upsilon \in [-\tau, x], \\ \Theta, & \upsilon \in (x, 0]. \end{cases}$$

Thus

$$\left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\right\| = \begin{cases} \left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\right\|, & \upsilon \in [-\tau, x], \\ 0, & \upsilon \in (x, 0]. \end{cases}$$

Therefore, from Lemma 2.4, we have

$$\begin{aligned}
\left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\right\| &\le (x-\upsilon)\,E_\alpha\left(\frac{\|B\|}{\alpha+1}, x-\upsilon\right) \\
&\le (x+\tau)\,E_\alpha\left(\frac{\|B\|}{\alpha+1}, x+\tau\right),
\end{aligned} \tag{21}$$

for $-\tau \le \upsilon \le 0$, $x \in W$, and since $E_\alpha\left(\frac{\|B\|}{\alpha+1}, x-\upsilon\right)$ is increasing function when $x \ge \upsilon$. From Eq. (21), we get

$$\int_{-\tau}^{0}\left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^\alpha\right)\right\| d\upsilon \le \tau\,(x+\tau)\,E_\alpha\left(\frac{\|B\|}{\alpha+1}, x+\tau\right). \tag{22}$$

From Lemma 2.4, we have

$$\begin{aligned}
&\int_0^x \left\|\mathcal{M}_{\tau,\alpha}\left(B(x-\tau-\upsilon)^{\alpha}\right)\right\| \upsilon^{\alpha-2} d\upsilon \\
&\leq \int_0^x (x-\upsilon) E_{\alpha}\left(\frac{\|B\|}{\alpha+1}, x-\upsilon\right) \upsilon^{\alpha-2} d\upsilon \\
&\leq E_{\alpha}\left(\frac{\|B\|}{\alpha+1}, x\right) \int_0^x (x-\upsilon)\upsilon^{\alpha-2} d\upsilon \\
&= \frac{x^{\alpha}}{\alpha(\alpha-1)} E_{\alpha}\left(\frac{\|B\|}{\alpha+1}, x\right).
\end{aligned} \tag{23}$$

From Eqs. (20), (22) and (23), we get

$$\begin{aligned}
\|y(x)\| \leq \delta E_{\alpha}\left(\frac{\|B\|}{\alpha-1}, x\right) + \delta(x+\tau) E_{\alpha}\left(\frac{\|B\|}{\alpha+1}, x+\tau\right) \\
+ \delta\tau(x+\tau) E_{\alpha}\left(\frac{\|B\|}{\alpha+1}, x+\tau\right) + \frac{\|f\|_C}{\alpha(\alpha-1)} x^{\alpha} E_{\alpha}\left(\frac{\|B\|}{\alpha+1}, x\right),
\end{aligned} \tag{24}$$

for all $x \in W$. Combining Eq. (19) with Eq. (24), we obtain $\|y(x)\| < \beta$ for all $x \in W$. This completes the proof.

Corollary 4.1. Let $\alpha = 2$ in Eq. (1). Then the system

$$y''(x) = -By(x-\tau) + f(x), \quad \text{for} \quad x \geq 0, \tau > 0,$$

$$y(x) \equiv \psi(x), y'(x) \equiv \psi'(x) \quad \text{for} \quad -\tau \leq x \leq 0,$$

is finite time stable with respect to $\{0, W, \tau, \delta, \beta\}$, $\delta < \beta$ if

$$E_2\left(\frac{\|B\|}{3}, L+\tau\right) < \frac{\beta - \delta E_2(\|B\|, L) - \frac{\|f\|_C}{2} L^2 E_2\left(\frac{\|B\|}{3}, L\right)}{\delta(L+\tau)(\tau+1)}.$$

Remark 4.1. Let $\alpha = 2$, $B = B^2$ in Eq. (1) such that the matrix B is a nonsingular $n \times n$ matrix. Then the representation of solution Eq. (18) coincides with the conclusion of Theorems 1 and 2 in [12], which leads to the same of the finite time stability results in [27].

5 An Example

Consider the conformable delay differential equations

$$\begin{aligned}
&\left(\mathfrak{D}_0^{1.8} y\right)(x) = -By(x-0.5) + f(x), \; x \in [0,1], \\
&\psi(x) = \left(0.1x^2, 0.2x\right)^T, \psi'(x) = (0.2x, 0.2)^T, \psi''(x) = (0.2, 0)^T, -0.5 \leq x \leq 0,
\end{aligned} \tag{25}$$

where

$$\alpha = 1.8, \; \tau = 0.5, \; \mathrm{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \; \mathrm{f}(x) = \begin{pmatrix} x^{1/5} \\ 2x^{1/5} \end{pmatrix}.$$

From Theorems 3.1 and 3.2, for all $0 \le x \le 1$, and through a basic calculation, we can obtain

$$\begin{aligned} y(x) &= \begin{pmatrix} 0.025\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) \\ -0.1\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) \end{pmatrix} + \begin{pmatrix} -0.1\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\ 0.2\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0.2\int_{-0.5}^{0}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \int_{0}^{x}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon \\ 2\int_{0}^{x}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon \end{pmatrix} = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} y_1(x) &= 0.025\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) - 0.1\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\ &\quad + 0.2\int_{-0.5}^{0}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon \\ &\quad + \int_{0}^{x}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon, \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= -0.1\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) + 0.2\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\ &\quad + 2\int_{0}^{x}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon, \end{aligned}$$

where

$$\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) = \begin{cases} 1, & -0.5 \le x < 0, \\ 1 - \dfrac{25}{18}x^{1.8}, & 0 \le x < 0.5, \\ 1 - \dfrac{25}{18}x^{1.8} + \dfrac{625}{2106}(x-0.5)^{3.6}, & 0.5 \le x < 1, \end{cases}$$

and

$$\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) = \begin{cases} (x+0.5), & -0.5 \le x < 0, \\ (x+0.5) - \dfrac{25}{63}x^{2.8}, & 0 \le x < 0.5, \\ (x+0.5) - \dfrac{25}{63}x^{2.8} + \dfrac{625}{13041}(x-0.5)^{4.6}, & 0.5 \le x < 1. \end{cases}$$

Thus the explicit solutions of Eq. (25) are

$$\begin{aligned} y_1(x) &= 0.025\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) - 0.1\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\ &\quad + 0.2\int_{-0.5}^{x-0.5}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon \\ &\quad + 0.2\int_{x-0.5}^{0}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon \\ &\quad + \int_{0}^{x}\mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right)d\upsilon, \end{aligned}$$

$$
\begin{aligned}
y_2(x) &= -0.1\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) + 0.2\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\
&\quad + 2\int_0^x \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon,
\end{aligned}
$$

where $0 \le x \le 0.5$, which implies that

$$
y_1(x) = -\frac{25}{1197}x^{3.8} + \frac{5}{126}x^{2.8} + \frac{1}{2}x^2 - \frac{5}{144}x^{1.8},
$$

$$
y_2(x) = -\frac{5}{63}x^{2.8} + x^2 + \frac{5}{36}x^{1.8} + \frac{1}{5}x,
$$

and

$$
\begin{aligned}
y_1(x) &= 0.025\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) - 0.1\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\
&\quad + 0.2\int_{-0.5}^{x-1} \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon \\
&\quad + 0.2\int_{x-1}^{0} \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon \\
&\quad + \int_0^{x-0.5} \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon \\
&\quad + \int_{x-0.5}^{x} \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon, \\
y_2(x) &= -0.1\mathcal{H}_{0.5,1.8}\left(2x^{1.8}\right) + 0.2\mathcal{M}_{0.5,1.8}\left(2x^{1.8}\right) \\
&\quad + 2\int_0^{x-0.5} \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon \\
&\quad + 2\int_{x-0.5}^{x} \mathcal{M}_{0.5,1.8}\left(2(x-0.5-\upsilon)^{1.8}\right) d\upsilon,
\end{aligned}
$$

where $0.5 \le x \le 1$, which implies that

$$
\begin{aligned}
y_1(x) &= \frac{625}{365148}(x-0.5)^{5.6} - \frac{125}{26082}(x-0.5)^{4.6} - \frac{100}{1197}(x-0.5)^{3.8} \\
&\quad + \frac{125}{16848}(x-0.5)^{3.6} - \frac{25}{1197}x^{3.8} + \frac{5}{126}x^{2.8} + \frac{1}{2}x^2 - \frac{5}{144}x^{1.8}, \\
y_2(x) &= \frac{125}{13041}(x-0.5)^{4.6} - \frac{250}{1197}(x-0.5)^{3.8} \\
&\quad - \frac{125}{4212}(x-0.5)^{3.6} - \frac{5}{63}x^{2.8} + x^2 + \frac{5}{36}x^{1.8} + \frac{1}{5}x.
\end{aligned}
$$

By calculating we obtain $\eta = \max\left\{\|\psi\|_C, \|\psi'\|_C, \|\psi''\|_C\right\} = 0.3$, $\|B\| = 2$, $\|f\|_C = 3$, $E_\alpha\left(\frac{2}{0.8}, L\right) = 4.0104$, $E_\alpha\left(\frac{2}{2.8}, L+0.5\right) = 2.278$, $E_\alpha\left(\frac{2}{2.8}, L\right) = 1.4871$, then we set $\delta = 0.31 > 0.3 = \eta$. Fig. 1 shows the state $y(x)$ and the norm $\|y(x)\|$ of Eq. (25). Now Theorem 4.1 implies that $\|y(x)\| \le 5.930254$, we just take $\beta = 5.9303$, which implies that $\|y(x)\| < \beta$ and Eq. (25) is finite time stable.

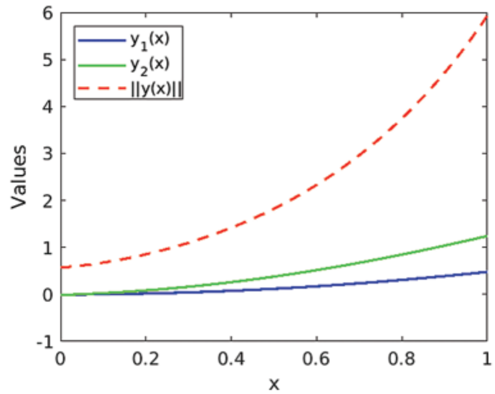

Figure 1: The state $y(x)$ and $||y(x)||$ of Eq. (25)

6 Conclusion

In this work, using new conformable delayed matrix functions, we derived explicit solutions of linear conformable fractional delay systems of order $\alpha \in (1, 2]$, which extend and improve the corresponding and existing ones in [12,13] in the case of $\alpha = 2$ without any restrictions on the matrix coefficient of the linear part, by removing the condition that B is a nonsingular matrix and replacing the matrix coefficient of the linear part B^2 in [12] by an arbitrary, not necessarily squared, matrix. In addition, using the formula of general solutions and a norm estimation of the conformable delayed matrix functions, we established some sufficient conditions for the finite time stability results, which extend and improve the existing ones in [27] in the case of $\alpha = 2$. Ultimately, an illustrative example was given to show the validity of the proposed results.

Following the topic of this paper, we outline some possible next research directions. The first direction will include applying the results of this paper on control problems for conformable fractional delay systems of order $\alpha \in (1, 2]$. The second direction is to consider the explicit solutions of linear conformable fractional delay systems of the form

$$\begin{aligned} &\mathfrak{D}_0^{\alpha}\left(\mathfrak{D}_0^{\alpha}y\right)(x) = -By\left(x-\tau\right), \quad \text{for} \quad x \geq 0, \tau > 0, \\ &y\left(x\right) \equiv \psi\left(x\right), y'\left(x\right) \equiv \psi'\left(x\right) \quad \text{for} \quad -\tau \leq x \leq 0, 0 < \alpha \leq 1, \end{aligned}$$

which lead to new results on stability and control problems. Depending on these results and delayed arguments, we will try to prove a generalized Lyapunov-type inequality for the conformable and sequential conformable boundary value problems

$$\begin{aligned} &\left(\mathfrak{D}_a^{\alpha}y\right)(x) = -By\left(x-\tau\right), \quad \text{for} \quad x \in (\mathrm{a}, \mathrm{b}), \alpha \in (1, 2] \\ &\qquad y\left(a\right) \equiv y\left(b\right) = \Theta, -\tau \leq x \leq 0, \\ &\left(\mathfrak{D}_a^{2\alpha}y\right)(x) = -By\left(x-\tau\right), \quad \text{for} \quad x \in (\mathrm{a}, \mathrm{b}), \alpha \in \left(\frac{1}{2}, 1\right] \\ &\qquad y\left(a\right) \equiv y\left(b\right) = \Theta, -\tau \leq x \leq 0, \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{D}_a^{\alpha_1}\left(\mathfrak{D}_a^{\alpha_2}y\right)(x) = -By\left(x-\tau\right), \quad \text{for} \quad x \in (\mathrm{a}, \mathrm{b}), \alpha_1, \alpha_2 \in (0, 1] \\ &\quad y\left(a\right) \equiv y\left(b\right) = \Theta \quad \text{for} \quad -\tau \leq x \leq 0, 1 < \alpha_1 + \alpha_2 \leq 2, \end{aligned}$$

which leads to new results on the conformable Sturm-Liouville eigenvalue problem.

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