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Article

Extended Rama Distribution: Properties and Applications

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Abstract: In this paper, the Rama distribution (RD) is considered, and a new model called extended Rama distribution (ERD) is suggested. The new model involves the sum of two independent Rama distributed random variables. The probability density function (pdf) and cumulative distribution function (cdf) are obtained and analyzed. It is found that the new model is skewed to the right. Several mathematical and statistical properties are derived and proved. The properties studied include moments, coefficient of variation, coefficient of skewness, coefficient of kurtosis and moment generating function. Some simulations are undertaken to illustrate the behavior of these properties. In addition, the reliability analysis of the distribution is investigated through the hazard rate function, reversed hazard rate function and odds function. The parameter of the distribution is estimated based on the maximum likelihood method. The distributions of order statistics for ERD are also presented. The performance of the suggested model is compared with several other lifetime distributions based on some goodness of fit tests on a real dataset. It turns out that the suggested model is more flexible than its competitors considered in this study, for modeling real lifetime data.

Keywords: Coefficient of kurtosis; coefficient of skewness; order statistics; two independent Rama random variables; Maximum likelihood estimation; Rama distribution

1 Introduction

Recently, some studies on the distribution of sum of random variables have acquired some significant importance in various branches of science for fitting real data. Many authors have studied the distribution of the sum of random variables and their applications based on various base distributions, to suggest new flexible distributions. As an example, [1] suggested a power length-biased Suja distribution, whereas [2] derived the distribution for the sums of uniformly distributed random variables. Moreover, [3] introduced the distribution of the sum of mixed independent random variables pertaining to special functions. Also, [4] proposed the sum and difference of two squared correlated Nakagami variates which are in connection with the McKay distribution. In contrast, the sum of t and Gaussian random variables is suggested by [5], while [6] proposed the sum of independent gamma random variables. Reference [7] studied the sum of independent gamma random variables, while generalizations of two-sided power distributions and their convolution are introduced by [8]. Besides, [9] derived the distribution of the mixed sum of independent random variables where one of them is associated with the H-function. On the other hand, [10] obtained the distribution of the sum of mixed independent random variables pertaining to certain special functions, while [11] proposed the weighted Suja distribution and its applications in engineering. Moreover, [12] suggested a Darna distribution as a mixture of exponential and gamma distributions. Reference [13] proposed a power size biased two-parameter Akash distribution, and [14] suggested a generalization of the new Weibull-Pareto distribution. In 2019, [15] introduced a transmuted Ishita distribution and [16] proposed a Marshall-Olkin length-biased exponential distribution.

In this paper, we modified the Rama distribution, which was firstly suggested by [17], while considering the sum of two independent variables in order to propose a new lifetime distribution, named extended Rama distribution. The probability density function (pdf) of the Rama distribution is given by

$$f(x;\theta) = \frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}, x>0, \theta>0, \tag{1}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x;\theta) = 1-\left(1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right)e^{-\theta x}; x>0, \theta>0. \tag{2}$$

The rest of the paper is organized as follows. In Section 2, the new distribution is presented in terms of its functions and shapes. Some statistical properties are given in Section 3, while in Section 4, the parameter of the distribution is estimated using the maximum likelihood method. Distributions of order statistics based on samples selected from ERD are presented in Section 5. Moreover, the reliability analysis based on ERD is given in Section 6. An application using a real data set is provided in Section 7. Finally, the paper is concluded in Section 8.

1.1 The New Model

Let X and Y be two independent random variables defined on the interval $[0,\infty)$ follows the Rama density with parameter θ. Suppose that the density of the random variable $Z = X + Y$ indicates the distribution of interest, and f_X, f_Y and f_Z denote the densities for random variables X, Y and Z, respectively. Therefore,

$$f_X(x) = f_Y(x) = \frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}, x>0, \theta>0, \tag{3}$$

and

$$f_Z(x) = \int_0^\infty f(x-t).f(t)dt. \tag{4}$$

Hence, a random variable X is said to follow the ERD if its pdf is given by

$$f(x;\theta) = \frac{\theta^8}{\left(6+\theta^3\right)^2}\left(x+\frac{x^4}{2}+\frac{x^7}{140}\right)\mathrm{e}^{-\theta x},\ x>0, \theta>0. \tag{5}$$

It is easy to show and validate Eq. (5) as follows:

$$
\begin{aligned}
f(x;\theta) &= \int_{-\infty}^{\infty} f(x-t).f(t)dt \\
&= \int_{0}^{x} f(1+(x-t))f(1+t)dt \\
&= \int_{0}^{x} \frac{\theta^4}{6+\theta^3}(1+(x-t)^3 e^{-\theta(x-t)} \frac{\theta^4}{6+\theta^3}\left(1+t^3\right)e^{-\theta x}dt \\
&= \frac{\theta^8}{\left(6+\theta^3\right)^2} e^{-\theta x} \int_{0}^{x} \left(x^3+3t^2x-3tx^2-t^3\right)\left(1+t^3\right)dt \\
&= \frac{\theta^8}{\left(6+\theta^3\right)^2} e^{-\theta x} \int_{0}^{x} \left(x^3t^3+3t^5x-3t^4x^2-t^6\right)dt \\
&= \frac{\theta^8}{\left(6+\theta^3\right)^2}\left(x+\frac{x^4}{2}+\frac{x^7}{140}\right)e^{-\theta x}.
\end{aligned}
$$

It is of interest to note here that

$$
\begin{aligned}
\int_{0}^{\infty} f(x;\theta)dx &= \int_{0}^{\infty} \frac{\theta^8}{\left(6+\theta^3\right)^2}\left(x+\frac{x^4}{2}+\frac{x^7}{140}\right)\mathrm{e}^{-x\theta}dx \\
&= \frac{\theta^8}{\left(6+\theta^3\right)^2}\int_{0}^{\infty}\left(x+\frac{x^4}{2}+\frac{x^7}{140}\right)\mathrm{e}^{-x\theta}dx \\
&= \frac{\theta^8}{\left(6+\theta^3\right)^2}\left(\frac{1}{\theta^2}+\frac{12}{\theta^5}+\frac{36}{\theta^8}\right) \\
&= \frac{\theta^8}{\left(6+\theta^3\right)^2}\left(\frac{\left(6+\theta^3\right)^2}{\theta^8}\right) = 1.
\end{aligned}
$$

Theorem 1: Let X be a random variable follows the ERD with parameter θ. The cumulative distribution function of X is defined as

$$
F(x;\theta) = 1 - \left(\frac{x^2\theta^2\left[42x^3\theta^3+7x^4\theta^4+x^5\theta^5+\left(3+\theta^3\right)\left(840+280x\theta+70x^2\theta^2\right)\right]}{140\left(6+\theta^3\right)^2}+1+x\theta\right)\mathrm{e}^{-x\theta}. \tag{6}
$$

Proof: The cdf of the ERD can be derived as follows:

$$F(x;\theta) = P(X \le x)$$
$$= \frac{\theta^8}{(6+\theta^3)^2} \int_0^x \left(y + \frac{y^4}{2} + \frac{y^7}{140} \right) e^{-y\theta} dy$$
$$= 1 - \frac{\left(x^2\theta^2 \left[42x^3\theta^3 + 7x^4\theta^4 + x^5\theta^5 + (3+\theta^3)(840 + 280x\theta + 70x^2\theta^2) \right] + 140(6+\theta^3)^2 (1+x\theta) \right)}{140(6+\theta^3)^2} e^{-x\theta}$$
$$= 1 - \left(\frac{x^2\theta^2 \left[42x^3\theta^3 + 7x^4\theta^4 + x^5\theta^5 + (3+\theta^3)(840 + 280x\theta + 70x^2\theta^2) \right]}{140(6+\theta^3)^2} + 1 + x\theta \right) e^{-x\theta}.$$

To study the behavior of the ERD, we consider $x \in (0, 2]$. In Fig. 1, the plots of the pdf and cdf of the ERD for various values of the parameter of the distribution are presented.

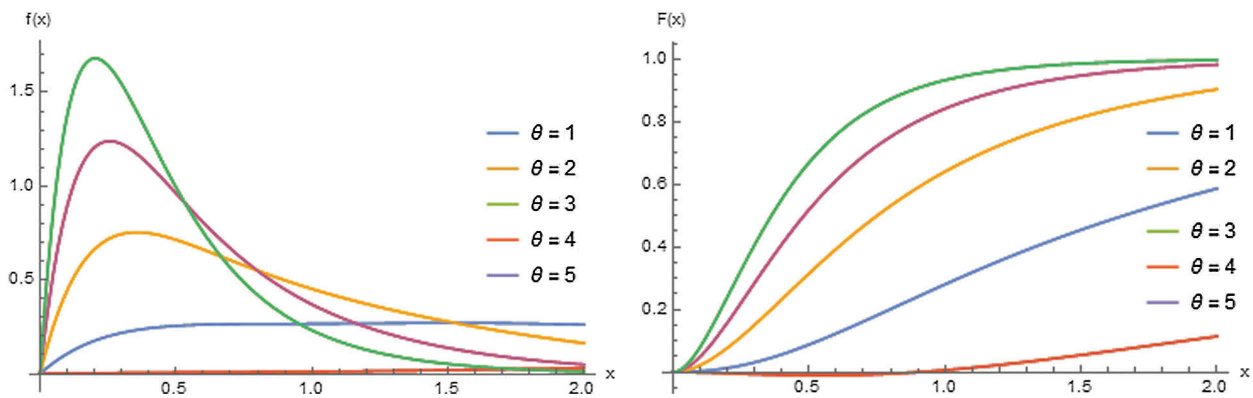

Figure 1: The pdf and cdf of ERD for θ = 1, 2, 3, 4, 5

Referring to Fig. 1, it is clear that ERD is asymmetric and skewed to the right over the interval (0, 2]. The degree of the skewness depends on the parameter value.

2 Some Properties of ERD

This section presents the r^{th} moment, mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis for the ERD. Some numerical calculations for these properties are also provided.

2.1 Moments

Theorem 2: Let $X \sim f(x;\theta)$. Then, the r^{th} moment of X is given by

$$E(X^r) = \frac{\theta^{-r}\left(70\theta^3\left(2\theta^3\Gamma(r+2) + \Gamma(r+5)\right) + \Gamma(r+8)\right)}{140(\theta^3+6)^2}, \theta > 0;\ r = 1,2,3,\ldots \quad (7)$$

Proof: The r^{th} moment of the ERD can be obtained by

$$\begin{aligned}
\mu'_r &= E(X^r) \\
&= \frac{\theta^8}{(6+\theta^3)^2}\int_0^\infty x^r\left(x+\frac{x^4}{2}+\frac{x^7}{140}\right)\mathrm{e}^{-x\theta}dx \\
&= \frac{\theta^8}{(6+\theta^3)^2}\left[\int_0^\infty x^{r+1}e^{-\theta x}dx+\int_0^\infty \frac{x^{4+r}}{2}e^{-\theta x}dx+\int_0^\infty \frac{x^{7+r}}{140}e^{-\theta x}dx\right] \\
&= \frac{\theta^8}{(6+\theta^3)^2}\left[\theta^{-r-2}\Gamma(r+2)+\frac{1}{2}\theta^{-r-5}\Gamma(r+5)+\frac{1}{140}\theta^{-r-8}\Gamma(r+8)\right] \\
&= \frac{\theta^{-r}\left(70\theta^3\left(2\theta^3\Gamma(r+2)+\Gamma(r+5)\right)+\Gamma(r+8)\right)}{140\left(\theta^3+6\right)^2}, \theta>0;\ r=1,2,3,\ldots
\end{aligned}$$

Based on Eq.7, we can obtain the first four moments of the ERD as

$$\mu'_1=\frac{48+2\theta^3}{6\theta+\theta^4}, \mu'_2=\frac{6\left(432+60\theta^3+\theta^6\right)}{\theta^2\left(6+\theta^3\right)^2}, \mu'_3=\frac{24\left(1080+105\theta^3+\theta^6\right)}{\theta^3\left(6+\theta^3\right)^2} \text{ and } \mu'_4=\frac{120\left(2376+168\theta^3+\theta^6\right)}{\theta^4\left(6+\theta^3\right)^2}, \tag{8}$$

respectively. Therefore, the variance of the ERD is

$$\sigma^2=\frac{2\left(\theta^6+84\theta^3+144\right)}{\theta^2\left(\theta^3+6\right)^2}. \tag{9}$$

2.2 The Coefficient of Skewness

The coefficient of skewness determines the degree of skewness of a distribution and for the ERD it is given by

$$Sk=\frac{\mu'_3-3\mu'_1\sigma^2-(\mu'_1)^3}{\sigma^3}=\frac{\sqrt{2}(\theta^9+198\theta^6+324\theta^3+864)}{\theta^3(\theta^3+6)^3\left(\dfrac{\theta^6+84\theta^3+144}{\theta^2(\theta^3+6)^2}\right)^{3/2}}. \tag{10}$$

2.3 The Coefficient of Kurtosis

The coefficient of variation of the ERD is defined as

$$ku=\frac{\mu_4-4\mu_1\mu_3+6{\mu_1}^2\sigma^2+3{\mu_1}^4}{\sigma^4}=\frac{6\left(\theta^3+6\right)^2\left(\theta^6+264\theta^3+360\right)}{\left(\theta^6+84\theta^3+144\right)^2}. \tag{11}$$

2.4 The Coefficient of Variation

The coefficient of variation of the ERD is defined as

$$Cv=\frac{\sigma}{\mu}=\frac{\sqrt{\theta^6+84\theta^3+144}}{\sqrt{2}\left(\theta^3+24\right)}. \tag{12}$$

To investigate the behavior of these measures, we calculate the values of $\mu,\ \sigma,\ Cv,\ Sk$ and Ku for the ERD for various values ofθ. The results are presented in Tab. 1.

Table 1: The values of $\mu,\ \sigma,\ Cv,\ Sk$ and Ku for the ERD for different values of θ

θ	μ	σ	Cv	Sk	Ku
0.1	79.9900	28.2878	0.3536	0.7068	3.7496
0.2	39.9601	14.1562	0.3543	0.7043	3.7470
0.3	26.5771	9.4595	0.3559	0.6978	3.7400
0.4	19.8417	7.1259	0.3591	0.6859	3.7266
0.5	15.7551	5.7402	0.3643	0.6683	3.7053
0.6	12.9858	4.8291	0.3719	0.6458	3.6756
0.7	10.9651	4.1885	0.3820	0.6209	3.6380
0.8	9.4103	3.7149	0.3948	0.5971	3.5944
0.9	8.1666	3.3497	0.4102	0.5777	3.5483
1	7.1429	3.0573	0.4280	0.5660	3.5039
2	2.2857	1.4983	0.6555	0.9015	3.8512
3	1.0303	0.8006	0.7771	1.3949	5.4420
4	0.6286	0.4953	0.7880	1.6419	6.7889
5	0.4550	0.3499	0.7692	1.6901	7.3090

Based on Tab. 1, it can be deduced that:

1. As the values of θ are increasing, the values of mean and standard deviation are decreasing.
2. As the values of θ are increasing, the values of the coefficient of variation are increasing.
3. The coefficient of skewness and coefficient of kurtosis are decreasing as the values of theta are increasing up to $\theta = 1$, then start increasing.

Theorem 3: The moment generating function of the ERD is given by

$$E(e^{tx}) = \frac{\theta^8\left(\theta^3 - t^3 + 3\theta t^2 - 3\theta^2 t + 6\right)^2}{\left(\theta^3+6\right)^2 (t-\theta)^8}. \tag{13}$$

Proof: The moment generating function can be derived as follows

$$\begin{aligned} E(e^{tx}) &= \int_0^\infty e^{tx}\frac{\theta^8}{(\theta^3+6)^2}\left(\frac{x^7}{140}+\frac{x^4}{2}+x\right)3e^{-}dx = \frac{\theta^8}{(\theta^3+6)^2}\int_0^\infty\left(\frac{x^7}{140}+\frac{x^4}{2}+x\right)e^{-(\theta-t)x}dx \\ &= \frac{\theta^8}{(\theta^3+6)^2}\left[\int_0^\infty \frac{x^7}{140}e^{-(\theta-t)x}dx + \int_0^\infty \frac{x^4}{2}e^{-(\theta-t)x}dx + \int_0^\infty xe^{-(\theta-t)x}dx\right] \\ &= \frac{\theta^8}{(\theta^3+6)^2}\left[\frac{36}{(t-\theta)^8}+\frac{12}{(\theta-t)^5}+\frac{1}{(t-\theta)^2}\right] = \frac{\theta^8(\theta^3-t^3+3\theta^2-3\theta^2 t+6)^6}{(\theta^3+6)^2(t-\theta)^8}. \end{aligned}$$

3 Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n chosen from the ERD with parameter θ. The maximum likelihood estimator for the ERD parameter can be derived as follows. The likelihood function of θ is given by

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^8}{\left(6+\theta^3\right)^2}\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right)\mathrm{e}^{-\theta x_i} = \left(\frac{\theta^8}{\left(6+\theta^3\right)^2}\right)^n \prod_{i=1}^{n}\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right)\mathrm{e}^{-\theta x_i}$$
$$= \frac{\theta^{8n}}{\left(6+\theta^3\right)^{2n}} e^{\sum_{i=1}^{n} -\theta x_i} \prod_{i=1}^{n}\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right) = \frac{\theta^{8n}}{\left(6+\theta^3\right)^{2n}} e^{-n\theta\bar{x}} \prod_{i=1}^{n}\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right)$$

.

Then, the log-likelihood function is

$$\begin{aligned} lnL(\theta) &= ln\left[\frac{\theta^{8^n}}{\left(6+\theta^3\right)^{2n}} e^{-n\theta\bar{x}} \prod_{i=1}^{n}\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right)\right] \\ &= 8nln(\theta) - 2nln\left(6+\theta^3\right) - n\theta\bar{x} + \sum_{i=1}^{n} ln\left(x_i + \frac{x_i^4}{2} + \frac{x_i^7}{140}\right). \end{aligned} \tag{14}$$

By taking the first derivative of Eq.14 with respect to θ and setting the results to 0, we obtain

$$\frac{\partial lnL(\theta)}{\partial\theta} = \frac{8n}{\theta} - \frac{6n\theta^2}{\theta^3+6} - n\bar{x} = 0. \tag{15}$$

Since there is no closed form solution for Eq. 15, i.e. the MLE of θ, denoted as $\hat{\theta}$, is the numerical solution for this equation.

4 Order Statistics

Assume that $X_1, X_2, \ldots X_n$ denote a random sample of size n from the ERD distribution. Also, suppose that $X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n:n)}$ denote the corresponding order statistics of the sample. The density function of the i^{th} order statistic $X_{(i:n)}$ for $1 \le i \le n$ is given by

$$f_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x). \tag{16}$$

By substituting the pdf and cdf of the ERD in Eq. (16), the pdf of $X_{(i:n)}$ is

$$f_{(i:n)}(x,\theta) = \frac{\left\{\begin{array}{l} n!\theta^8\left(\frac{x^7}{140}+\frac{x^4}{2}+x\right)\left(1-\left(\frac{\theta^2x^2\left(\begin{array}{c}\theta^5x^5+7\theta^4x^4+42\theta^3x^3\\ +\left(\theta^3+3\right)\left(\theta+70\theta^2x^2+1120\right)\end{array}\right)}{140\left(\theta^3+6\right)^2}+\theta x+1\right)e^{-\theta x}\right)^{i-1} \\ \left(\left(\frac{\theta^2x^2\left(\theta^5x^5+7\theta^4x^4+42\theta^3x^3+\left(\theta^3+3\right)\left(\theta+70\theta^2x^2+1120\right)\right)}{140\left(\theta^3+6\right)^2}+\theta x+1\right)e^{-\theta x}\right)^{n-i}\end{array}\right\}}{\left(\theta^3+6\right)^2(i-1)!(m-i)!}e^{-\theta x}. \tag{17}$$

From Eq. (17), the pdfs of the smallest order statistic $X_{(1:n)}$ and the largest order statistic $X_{(n:n)}$ are, respectively be given by

$$f_{(1:n)}(x;\theta) = \frac{n!\theta^8\left(\frac{x^7}{140} + \frac{x^4}{2} + x\right)\left(\frac{\theta^2x^2\left(\begin{array}{l}\theta^5x^5 + 7\theta^4x^4 + 42\theta^3x^3\\ +\left(\theta^3+3\right)\left(\theta + 70\theta^2x^2 + 1120\right)\end{array}\right)}{140\left(\theta^3+6\right)^2} + \theta x + 1\right)^{n-1}}{\left(\theta^3+6\right)^2(n-1)!}e^{-\theta nx}, \tag{18}$$

and

$$f_{(n:n)}(x;\theta) = \frac{n!\theta^8\left(\frac{x^7}{140} + \frac{x^4}{2} + x\right)\left(1 - e^{-\theta x}\left(\frac{\theta^2x^2\left(\begin{array}{l}\theta^5x^5 + 7\theta^4x^4 + 42\theta^3x^3\\ +\left(\theta^3+3\right)\left(\theta + 70\theta^2x^2 + 1120\right)\end{array}\right)}{140\left(\theta^3+6\right)^2} + \theta x + 1\right)\right)^{n-1}}{\left(\theta^3+6\right)^2(n-1)!}e^{-\theta x}. \tag{19}$$

5 Reliability Analysis

This section defines the reliability (survival) function, hazard rate function, reversed hazard rate function and odd function for the suggested model. The reliability function of the ERD is given by

$$\begin{aligned} R(x;\theta) &= 1 - F(x;\theta) \\ &= \left(\frac{\theta^2x^2\left(\theta^5x^5 + 7\theta^4x^4 + 42\theta^3x^3 + \left(\theta^3+3\right)\left(\theta + 70\theta^2x^2 + 840 + 280\right)\right)}{140\left(\theta^3+6\right)^2} + (\theta x + 1)\right)e^{-\theta x}. \end{aligned} \tag{20}$$

On the other hand, the hazard rate function of the ERD is given by

$$\begin{aligned} H(x;\theta) &= \frac{f(x;\theta)}{1 - F(x;\theta)} \\ &= \frac{\theta^8\left(\frac{x^7}{140} + \frac{x^4}{2} + x\right)}{\left(\theta^3+6\right)^2\left(\frac{\theta^2x^2\left(\theta^5x^5 + 7\theta^4x^4 + 42\theta^3x^3 + \left(\theta^3+3\right)\left(\theta + 70\theta^2x^2 + 1120\right)\right)}{140\left(\theta^3+6\right)^2} + \theta x + 1\right)}. \end{aligned} \tag{21}$$

Meanwhile, the reversed hazard rate function for the ERD distribution is defined as

$$\begin{aligned} RH(x;\theta) &= \frac{f(x;\theta)}{F(x;\theta)} \\ &= \frac{\theta^8\left(\frac{x^7}{140} + \frac{x^4}{2} + x\right)e^{-\theta x}}{\left(\theta^3+6\right)^2\left(1 - e^{-\theta x}\left(\frac{\theta^2x^2\left(\begin{array}{l}\theta^5x^5 + 7\theta^4x^4 + 42\theta^3x^3\\ +\left(\theta^3+3\right)\left(\theta + 70\theta^2x^2 + 1120\right)\end{array}\right)}{140\left(\theta^3+6\right)^2} + \theta x + 1\right)\right)}. \end{aligned} \tag{22}$$

The odds function for ERD is defined as

$$O(x;\theta) = \frac{F(x;\theta)}{1 - F(x;\theta)} = \frac{e^{\theta x}}{\frac{\theta^2 x^2 \left(\theta^5 x^5 + 7\theta^4 x^4 + 42\theta^3 x^3 + \left(\theta^3 + 3\right)\left(\theta + 70\theta^2 x^2 + 1120\right)\right)}{140\left(\theta^3 + 6\right)^2} + \theta x + 1} - 1. \tag{23}$$

Figs. 2–4 reveal that the reliability and reversed hazard rate functions decrease as the values of x increase. In contrast, the hazard and odd functions are increasing with x for various values of the parameter θ.

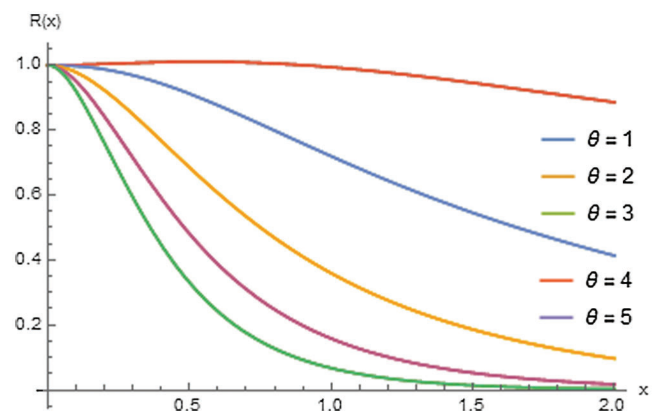

Figure 2: Reliability plots of the ERD for θ =1, 2, 3, 4, 5

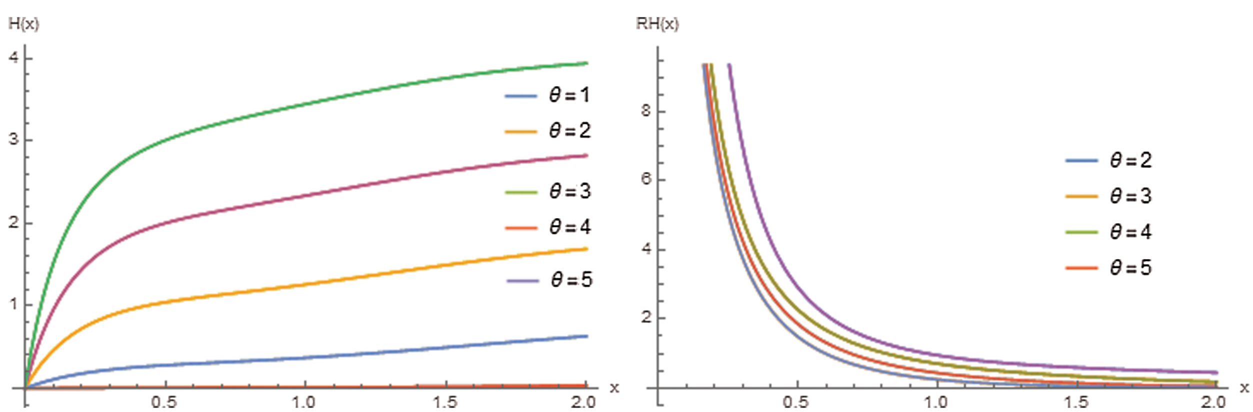

Figure 3: Plots of hazard and reversed hazard rate functions of the ERD for different values of θ

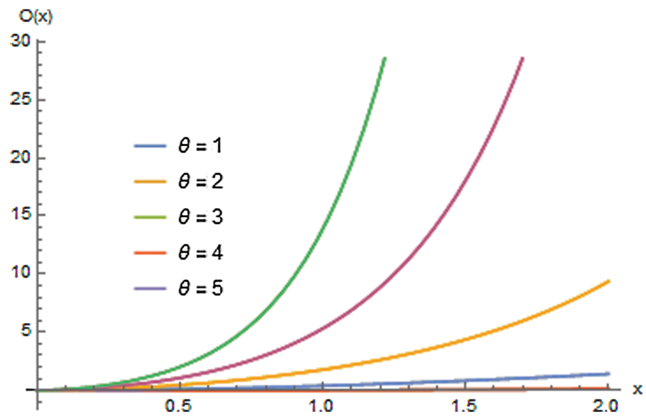

Figure 4: The plots of odds function of ERD for θ =1, 2, 3, 4, 5

6 Practical Illustration

This section compares the performance of ERD with Rama distribution, exponential distribution, Rani distribution and Maxwell length-biased distribution for fitting a real data set. The probability density functions for these distributions are as follows:

a. Rama distribution (RD) [17]:

$$f(x;\theta) = \frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}, x>0, \theta>0. \tag{24}$$

b. Exponential distribution (ED):

$$f(x;\lambda) = \lambda e^{-\lambda x}, \lambda>0. \tag{25}$$

c. Rani distribution (RND) [18]:

$$f(x;\theta) = \frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x}, \theta>0. \tag{26}$$

d. Maxwell length-biased distribution (MLBD) [19]:

$$f(x;\alpha) = \frac{x^3}{2\alpha^4}e^{-\frac{x^2}{2\alpha^2}}, x>0, \alpha>0. \tag{27}$$

The dataset explains the strength of the aircraft window glass as recorded by [20], which is given as follows:

Data: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78,27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08,37.09, 39.58, 44.045, 45.29, 45.381.

For comparison, we consider the Akaike Information Criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (KS) test statistics for measuring the goodness of fit (GOF) of the different models to the data. Parameters of the models are estimated using maximum likelihood. These (GOF) measures are defined as follows:

$$AIC = -2LL + 2\kappa,$$

$$CAIC = -2LL + \frac{2\varphi n}{n - \varphi - 1},$$

$$HQIC = 2Log\{Log(n)[\varphi - 2LL]\},$$

$$BIC = -2LL + \varphi Log(n)$$

where φ is the number of parameters and n is the sample size. The Kolmogorov-Smirnov (KS) test is defined as $KS = Sup_n \left| F_n(x) - F(x) \right|$, where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \leq x}$ is the empirical distribution function and $F(x)$ is the cumulative distribution function. Generally, the smallest value of these particular measures indicates the respective model best fits the data. For assessing the goodness of fit for all the models considered, the measures are computed and the results are given in Tab. 2.

Table 2: The MLE, AIC, CAIC, BIC, HQIC and the p-value for modelling the data

Model	MLE	AIC	CAIC	BIC	HQIC	KS	p-value
ERD	0.2591	219.407	219.545	220.841	229.73	0.1574	0.3860
RD	0.1298	234.792	234.930	236.226	235.26	0.2538	0.0301
ED	0.0324	276.529	276.667	277.963	254.82	0.4584	1.77e-06
RND	0.1623	229.250	229.388	229.718	243.15	0.2233	0.0772
MLBD	0.1581	221.037	221.175	221.505	242.95	0.1896	0.1889

Based on Fig. 5, we notice that ERD provides the best fit for modelling the dataset. The p-value of the KS test is 0.39 which is greater than the 0.05 level of significance, indicating that ERD model is adequate for the data. In addition, the other measures of goodness of fit are found the smallest for the ERD, with the respective larger p-values, which further support that ERD as the best model.

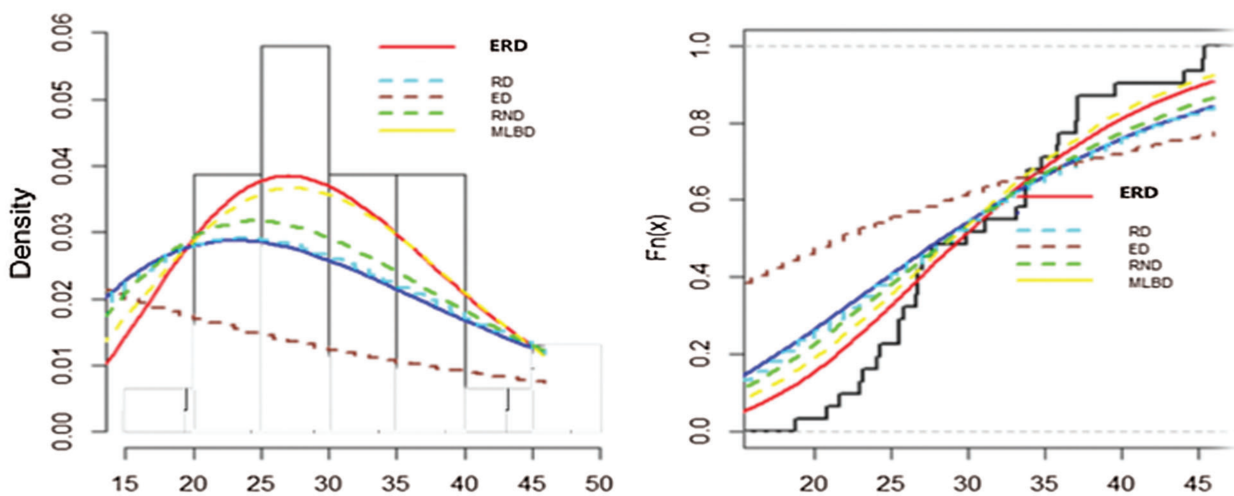

Figure 5: The histogram and empirical distribution functions of the fitted models to the data

7 Conclusions

In the present paper, an extended Rama distribution is proposed. Many mathematical and statistical properties of the new distribution are provided. The reliability, hazard rate, reversed hazard rate and odd functions of the ERD are also presented. The parameter of this distribution is obtained using MLE. In addition to a simulation study, the performance of ERD in fitting a real dataset is compared to several other models which are often used to describe the lifetime data based on several goodness of fit tests. It turns out that the ERD can be considered as a viable alternative model when modelling lifetime data. As for future work, one can estimate the parameter of the distribution using the ranked set sampling method [21–24].

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