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Reconstructing the Time-Dependent Thermal Coefficient in 2D Free Boundary Problems

M. J. Huntul*

Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia
*Corresponding Author: M. J. Huntul. Email: mhantool@jazanu.edu.sa
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Abstract: The inverse problem of reconstructing the time-dependent thermal conductivity and free boundary coefficients along with the temperature in a two-dimensional parabolic equation with initial and boundary conditions and additional measurements is, for the first time, numerically investigated. This inverse problem appears extensively in the modelling of various phenomena in engineering and physics. For instance, steel annealing, vacuum-arc welding, fusion welding, continuous casting, metallurgy, aircraft, oil and gas production during drilling and operation of wells. From literature we already know that this inverse problem has a unique solution. However, the problem is still ill-posed by being unstable to noise in the input data. For the numerical realization, we apply the alternating direction explicit method along with the Tikhonov regularization to find a stable and accurate numerical solution of finite differences. The root mean square error (*rmse*) values for various noise levels p for both smooth and non-smooth continuous time-dependent coefficients Examples are compared. The resulting nonlinear minimization problem is solved numerically using the MATLAB subroutine *lsqnonlin*. Both exact and numerically simulated noisy input data are inverted. Numerical results presented for two examples show the efficiency of the computational method and the accuracy and stability of the numerical solution even in the presence of noise in the input data.

Keywords: Inverse identification problem; two-dimensional parabolic equation; free boundary; Tikhonov regularization; nonlinear optimization

1 Introduction

Free boundary problems for the parabolic partial differential equations have significant applications in various fields of engineering, physics, chemistry, see [1–5] to mention only a few. A Stefan problem is a moving boundary value problem that concerns the distribution of heat in a period of transforming medium. The authors of [6] considered two fairly different free boundary problems of nonlinear diffusion. Chen et al. [7] recast the formulation of various shock diffraction/reflection problems as a free boundary problem. Shidfar et al. [8] studied the inverse moving boundary problem using the least-squares method. In [9], the authors investigated the numerical solution of inverse Stefan problems using the method of fundamental solutions.

Recently, the authors of [10] numerically solved the inverse heat problems of determining the time-dependent coefficients with free boundaries.

There is also an analysis of a one-dimensional inverse problem of reconstructing the timewise heat source with a moving boundary [11]. The authors in [12] estimated free boundary coming from two new scenarios, aggregation processes and nonlocal diffusion. Snitko [13,14], theoretically, and Huntul [15], numerically, investigated the inverse problem of determining the time-dependent reaction coefficient in a two-dimensional parabolic problem with a free boundary.

The challenge associated with free boundary problems arises from the finding that the solution domain is unknown. Only a few studies focus on the time-dependent free boundary in higher dimensions [16–19]. These papers are theoretical and very important because they present conditions for the unique solvability of the unknown coefficients.

This work examines the inverse problem to recover the time-dependent thermal conductivity coefficient and free boundaries from the heat flux and the nonlocal integral observations as over-specification conditions. The inverse problem presented in this paper has already been showed to be locally uniquely solvable by Barans'ka et al. [20], but no numerical determination has been realised so far, therefore, the main goal of this work is to attempt numerical solution of this problem.

The arrangement of this paper is systematized as follows. Section 2 describes the formulations of the inverse problem. The solution of the direct problem using the alternating direction explicit (ADE) method is presented in Section 3. The ADE direct solver is coupled with the Tikhonov regularization method in Section 4. In Section 5, computational results and discussions are presented. Finally, Section 6 highlights the conclusions.

2 Formulation of the Inverse Problem

Consider the inverse problem of reconstructing time-dependent thermal conductivity coefficient $a(t) > 0$, and the free boundaries $l(t) > 0$ and $h(t) > 0$ in the two-dimensional parabolic equation

$$u_t(x_1, x_2, t) = a(t)(u_{x_1x_1} + u_{x_2x_2}) + f(x_1, x_2, t), \quad (x_1, x_2, t) \in \Omega_T, \tag{1}$$

where $f(x_1, x_2, t)$ is a known heat source, $u = u(x_1, x_2, t)$ is the unknown temperature in the moving domain $\Omega_T := \{(x_1, x_2, t) \mid 0 < x_1 < l(t), 0 < x_2 < h(t), 0 < t < T < \infty\}$, with the initial condition

$$u(x_1, x_2, 0) = \varphi(x_1, x_2), \quad (x_1, x_2) \in [0, l_0] \times [0, h_0], \tag{2}$$

where $l_0 = l(0)$ and $h_0 = h(0)$ are given positive numbers, the Dirichlet boundary conditions

$$u(0, x_2, t) = \mu_1(x_2, t), \quad u(l(t), x_2, t) = \mu_2(x_2, t), \quad x_2 \in [0, h(t)], \quad t \in [0, T], \tag{3}$$

$$u(x_1, 0, t) = \mu_3(x_1, t), \quad u(x_1, h(t), t) = \mu_4(x_1, t), \quad x_1 \in [0, l(t)], \quad t \in [0, T], \tag{4}$$

and the over-specification conditions

$$a(t)u_{x_1}(0, Y_0, t) = \mu_5(t), \quad t \in [0, T], \tag{5}$$

where Y_0 is belong to the interval $Y_0 \in (0, h(t))$,

$$\int_0^{l(t)} \int_0^{h(t)} u(x_1, x_2, t)dx_2dx_1 = \mu_6(t), \quad t \in [0, T], \tag{6}$$

$$\int_0^{l(t)}\int_0^{h(t)} x_2 u(x_1, x_2, t)dx_2 dx_1 = \mu_7(t), \quad t \in [0, T], \tag{7}$$

where φ and μ_i for $i = \overline{1, 7}$ are given functions. The function $\mu_5(t)$ in Eq. (5) represents heat flux boundary condition. The function $\mu_6(t)$ in Eq. (6) corresponds to the specification of mass/energy, [21,22], whilst the function $\mu_7(t)$ in Eq. (7) represents the first-order moment specification, [23].

Introducing the new variables $y_1 = x_1/l(t)$ and $y_2 = x_2/h(t)$, see [20], we recast the problem given by Eqs. (1) to (7) into the problem given below for the unknowns $l(t)$, $h(t)$, $a(t)$ and $v(y_1, y_2, t) := u(y_1 l(t), y_2 h(t), t)$, namely,

$$v_t(y_1, y_2, t) = a(t)\left(\frac{1}{l^2(t)} v_{y_1 y_1} + \frac{1}{h^2(t)} v_{y_2 y_2}\right) + \frac{y_1 l'(t)}{l(t)} v_{y_1} + \frac{y_2 h'(t)}{h(t)} v_{y_2}$$
$$+ f(y_1 l(t), y_2 h(t), t), \quad (y_1, y_2, t) \in Q_T, \tag{8}$$

in the fixed domain $Q_T = \{(y_1, y_2, t) \mid 0 < y_1 < 1, 0 < y_2 < 1, 0 < t < T\}$,

$$v(y_1, y_2, 0) = \varphi(y_1 l_0, y_2 h_0), \quad (y_1, y_2) \in [0, 1] \times [0, 1], \tag{9}$$

$$v(0, y_2, t) = \mu_1(y_2 h(t), t), \quad v(1, y_2, t) = \mu_2(y_2 h(t), t), \quad y_2 \in [0, 1], \quad t \in [0, T], \tag{10}$$

$$v(y_1, 0, t) = \mu_3(y_1 l(t), t), \quad v(y_1, 1, t) = \mu_4(y_1 l(t), t), \quad y_1 \in [0, 1], \quad t \in [0, T], \tag{11}$$

$$\frac{a(t)}{l(t)} v_{y_1}(0, Y_0 h^{-1}(t), t) = \mu_5(t), \quad t \in [0, T], \tag{12}$$

$$l(t)h(t)\int_0^1\int_0^1 v(y_1, y_2, t)dy_2 dy_1 = \mu_6(t), \quad t \in [0, T], \tag{13}$$

$$l(t)h^2(t)\int_0^1\int_0^1 y_2 v(y_1, y_2, t)dy_2 dy_1 = \mu_7(t), \quad t \in [0, T]. \tag{14}$$

The existence and uniqueness of solution of the inverse problem Eqs. (8) to (14) has been established in [20] and read as follows.

Theorem 1 *Suppose that the following assumptions are satisfied:*

$(A1)$ $\varphi \in C([0, \infty) \times [0, \infty))$, $\mu_i \in C([0, \infty) \times [0, T])$ *for* $i = \overline{1, 4}$, $f \in C([0,\infty) \times [0,\infty) \times [0, T])$;

$(A2)$ $0 < \varphi_0 \le \varphi(x_1, x_2) \le \varphi_1 < \infty$, $(x_1, x_2) \in [0, \infty) \times [0, \infty)$, $\mu_i(t) > 0$ *for* $i = \overline{5, 7}$,
$t \in [0, T]$, $0 < \mu_{i0} \le \mu_i(x_2, t) \le \mu_{i1} < \infty$ *for* $i = 1, 2$, $(x_2, t) \in [0, \infty) \times [0, T]$,
$0 < \mu_{i0} \le \mu_i(x_1, t) \le \mu_{i1} < \infty$ *for* $i = 3, 4$, $(x_1, t) \in [0, \infty) \times [0, T]$,
$0 \le f(x_1, x_2, t) \le f_1 < \infty$, $(x_1, x_2, t) \in [0, \infty) \times [0, \infty) \times [0, T]$,
$\mu_{ix_1}(x_1, t) > 0$, $(x_1, t) \in [0, \infty) \times [0, T]$ *for* $i = 3, 4$, $\varphi_{x_1}(x_1, x_2) > 0$,
$(x_1, x_2) \in [0, l_0] \times [0, h_0]$;

$(A3)$ $\varphi \in C^2([0, l_0] \times [0, h_0]$, $\mu_5 \in C[0, T]$, $\mu_i \in C^1[0, T]$ *for* $i = 6, 7$,
$\mu_i \in C^{2,1}([0, \infty) \times [0, T])$, $i = 1, 2$, $\mu_i \in C^{2,1}([0, \infty) \times [0, T])$ *for* $i = 3, 4$,

$f \in C^{1,0}([0,\infty) \times [0,\infty) \times [0,T])$;

$(A4)$ compatibility conditions of the zeroth and first orders.

Then, it is possible to indicate a time $T_0 \in (0, T]$, determined by the input data, such that there exists a solution $(l(t), h(t), a(t), v(y_1, y_2, t)) \in C^1[0, T_0] \times C^1[0, T_0] \times C[0, T_0] \times C^{2,1}(\overline{\Omega}_{T_0})$ with $l(t) > 0$, $h(t) > 0$, $a(t) > 0$ for $t \in [0, T_0]$, to problem Eqs. (8) to (14).

Theorem 2 *Suppose that the following conditions are fulfilled:*

$(A5)$ $0 < \varphi_0 \le \varphi(x_1, x_2) \le \varphi_1 < \infty$, $\mu_5(t) \neq 0$, $\mu_6(t) \neq 0$, $\mu_7(t) \neq 0$ *for* $t \in [0, T]$,
$\mu_2(x_2, t) \neq 0$, $(x_2, t) \in [0,\infty) \times [0, T]$, $\mu_4(x_1, t) \neq 0$, $(x_1, t) \in [0,\infty) \times [0, T]$;

$(A6)$ $f \in C^{1,0}([0,\infty) \times [0,\infty) \times [0,T])$, $\mu_{itx_2} \in C([0,\infty) \times [0,T])$ *for* $i = 1, 2$,
$\mu_{itx_1} \in C([0,\infty) \times [0,T])$ *for* $i = 3, 4$.

Then, it is possible to indicate a time $T_1 \in (0, T]$, determined by the input data, such that problem Eqs. (8) to (14) has at most one solution $(l(t), h(t), a(t), v(y_1, y_2, t)) \in C^1[0, T_1] \times C^1[0, T_1] \times C[0, T_1] \times C^{2,1}(\overline{\Omega}_{T_1})$ with $l(t) > 0$, $h(t) > 0$, $a(t) > 0$ for $t \in [0, T_1]$.

3 Numerical Solution of the Forward Problem

Consider the direct (forward) problem Eqs. (8) to (11). When $l(t)$, $h(t)$, $a(t)$, $f(x_1, x_2, t)$, $\mu_i(t)$, for $i = \overline{1, 4}$ and $\varphi(y_1 l_0, y_2 h_0)$ are given in the direct problem $v(y_1, y_2, t)$ is to be found along with the quantities of interest $\mu_i(t)$ for $i = \overline{5, 7}$. Denote $v(y_1, y_2, t_n) = v^n_{i,j}$, $l(t_n) = l_n$, $h(t_n) = h_n$, $a(t_n) = a_n$ and $f(y_1 l(t_n), y_2 h(t_n), t_n) = f^n_{i,j}$, where $y_{1i} = i\Delta y_1$, $y_{2j} = j\Delta y_2$, $t_n = n\Delta t$, $\Delta y_1 = 1/M_1$, $\Delta y_2 = 1/M_2$, $\Delta t = T/N$, $i = 0, 1, \ldots, M_1$, $j = 0, 1, \ldots, M_2$, $n = 0, 1, \ldots, N$.

The ADE method, [3,24], which is unconditionally stable, is described for solving numerically the direct problem Eqs. (8) to (11). Let $\tilde{v}^n_{i,j}$ and $\tilde{u}^n_{i,j}$ satisfy

$$\tilde{v}^{n+1}_{i,j} = A_n \tilde{v}^n_{i,j} + B_n\left(\tilde{v}^n_{i+1,j} + \tilde{v}^{n+1}_{i-1,j}\right) + C_n\left(\tilde{v}^n_{i,j+1} + \tilde{v}^{n+1}_{i,j-1}\right) + D_n\left(\tilde{v}^n_{i+1,j} - \tilde{v}^{n+1}_{i-1,j}\right) + E_n(\tilde{v}^n_{i,j+1} - \tilde{v}^{n+1}_{i,j-1}) + G^*_{i,j}, \quad i = \overline{1, M_1 - 1}, \quad j = \overline{1, M_2 - 1}, \quad n = \overline{0, N-1}, \tag{15}$$

$$\tilde{u}^{n+1}_{i,j} = A_n \tilde{u}^n_{i,j} + B_n\left(\tilde{u}^{n+1}_{i+1,j} + \tilde{u}^n_{i-1,j}\right) + C_n\left(\tilde{u}^{n+1}_{i,j+1} + \tilde{u}^n_{i,j-1}\right) + D_n\left(\tilde{u}^{n+1}_{i+1,j} - \tilde{u}^n_{i-1,j}\right) + E_n(\tilde{u}^{n+1}_{i,j+1} - \tilde{u}^n_{i,j-1}) + G^*_{i,j}, \quad i = \overline{M_1 - 1, 1}, \quad j = \overline{M_2 - 1, 1}, \quad n = \overline{0, N-1}, \tag{16}$$

where

$$A_n = \frac{1 - \lambda_n}{1 + \lambda_n}, \quad B_n = \frac{(\Delta t)\, a_n}{l_n^2 (\Delta y_1)^2 (1 + \lambda_n)}, \quad C_n = \frac{(\Delta t)\, a_n}{h_n^2 (\Delta y_2)^2 (1 + \lambda_n)},$$

$$D_n = \frac{(\Delta t)\, y_{1i} l_{n'}}{2 (\Delta y_1)(1 + \lambda_n)\, l_n}, \quad E_n = \frac{(\Delta t)\, y_{2j} h_{n'}}{2 (\Delta y_2)(1 + \lambda_n)\, h_n},$$

$$G^*_{i,j} = \frac{\Delta t}{2\left(1 + \lambda^n_{i,j}\right)}\left(f^{n+1}_{i,j} + f^n_{i,j}\right), \quad \lambda_n = \frac{(\Delta t)\, a_n}{l_n^2 (\Delta y_1)^2} + \frac{(\Delta t)\, a_n}{h_n^2 (\Delta y_2)^2}.$$

In the above expression, we approximate the derivatives of $l(t)$ and $h(t)$ as

$$l'_n := l'(t_n) \approx \frac{l(t_n) - l(t_{n-1})}{\Delta t} = \frac{l_n - l_{n-1}}{\Delta t}, \quad n = \overline{1, N}, \tag{17}$$

$$h'_n := h'(t_n) \approx \frac{h(t_n) - h(t_{n-1})}{\Delta t} = \frac{h_n - h_{n-1}}{\Delta t}, \quad n = \overline{1, N}. \tag{18}$$

Furthermore, let the $\tilde{v}^n_{i,j}$ and $\tilde{u}^n_{i,j}$ also satisfy the initial and boundary conditions Eqs. (9) to (11), namely,

$$\tilde{v}^0_{i,j} = \tilde{u}^0_{i,j} = \varphi(y_i l_0, y_j h_0), \quad i = \overline{0, M_1}, \quad j = \overline{0, M_2}, \tag{19}$$

$$\tilde{v}^n_{0,j} = \tilde{u}^n_{0,j} = \mu_1(y_{2j} h_n, t_n), \quad \tilde{v}^n_{M_1,j} = \tilde{u}^n_{M_1,j} = \mu_2(y_{2j} h_n, t_n), \quad j = \overline{0, M_2}, \quad n = \overline{1, N}, \tag{20}$$

$$\tilde{v}^n_{i,0} = \tilde{u}^n_{i,0} = \mu_3(y_{1i} l_n, t_n), \quad \tilde{v}^n_{i,M_2} = \tilde{u}^n_{i,M_2} = \mu_4(y_{1i} l_n, t_n), \quad i = \overline{0, M_1}, \quad n = \overline{1, N}. \tag{21}$$

Once $\tilde{v}^{n+1}_{i,j}$ and $\tilde{u}^{n+1}_{i,j}$ have been obtained, the solution for the direct problem Eqs. (8) to (11) is computed by

$$v^{n+1}_{i,j} = \frac{\tilde{v}^{n+1}_{i,j} + \tilde{u}^{n+1}_{i,j}}{2}. \tag{22}$$

The heat flux in Eq. (12) can be approximated using the second-order FDM as follows.

$$\mu_5(t) = \frac{a_n}{l_n}\left(\frac{4v(1, Y_0 h_n^{-1}, t_n) - v(2, Y_0 h_n^{-1}, t_n) - 3v(0, Y_0 h_n^{-1}, t_n)}{2\Delta y_1}\right), \quad n = \overline{1, N}. \tag{23}$$

The integrals in Eqs. (13) and (14) can be calculated using the trapezoidal rule as follows:

$$\begin{aligned} l(t_n) h(t_n) \int_0^1 \int_0^1 v(y_1, y_2, t_n)\, dy_2 dy_1 = \frac{l_n h_n}{4 M_1 M_2} [& v(0, 0, t_n) + v(1, 0, t_n) + v(0, 1, t_n) + v(1, 1, t_n) \\ & + 2\sum_{i=1}^{M_1-1} v(y_{1i}, 0, t_n) + 2\sum_{i=1}^{M_1-1} v(y_{1i}, 1, t_n) + 2\sum_{j=1}^{M_2-1} v(0, y_{2j}, t_n) \\ & + 2\sum_{j=1}^{M_2-1} v(1, y_{2j}, t_n) + 4\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1-1} v(y_{1i}, y_{2j}, t_n)], \quad n = \overline{1, N}, \end{aligned} \tag{24}$$

$$\begin{aligned} l(t_n) h^2(t_n) \int_0^1 \int_0^1 y_2 v(y_1, y_2, t_n)\, dy_2 dy_1 = \frac{l_n h_n^2}{4 M_1 M_2} [& y_2(0) v(0, 0, t_n) + y_2(0) v(1, 0, t_n) \\ & + y_2(1) v(0, 1, t_n) + y_2(1) v(1, 1, t_n) \\ & + 2\sum_{i=1}^{M_1-1} y_2(0) v(y_{1i}, 0, t_n) + 2\sum_{i=1}^{M_1-1} y_2(1) v(y_{1i}, 1, t_n) \end{aligned}$$

$$+2\sum_{j=1}^{M_2-1} y_{2j} v\left(0, y_{2j}, t_n\right) + 2\sum_{j=1}^{M_2-1} y_{2j} v\left(1, y_{2j}, t_n\right)$$

$$+4\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1-1} y_{2j} v\left(y_{1i}, y_{2j}, t_n\right)], \quad n=\overline{1, N}. \tag{25}$$

4 Numerical Solution of the Inverse Problem

In this section, our aim is to obtain stable and accurate reconstructions of the free boundary $l(t)$, $h(t)$, the thermal conductivity $a(t)$ together with the temperature $v(y_1, y_2, t)$, satisfying Eqs. (8) to (14). The inverse problem can be formulated as a nonlinear least-squares minimization of the least-squares objective function given by

$$\mathbb{F}(l, h, a) = \left\| \frac{a(t)}{l(t)} v_{y_2}\left(0, Y_0 h^{-1}(t), t\right) - \mu_5(t) \right\|^2 + \left\| l(t)\, h(t) \int_0^1 \int_0^1 v(y_1, y_2, t)\, dy_2 dy_1 - \mu_6(t) \right\|^2 + \left\| l(t) h^2(t) \int_0^1 \int_0^1 y_2 v(y_1, y_2, t) dy_2 dy_1 - \mu_7(t) \right\|^2, \tag{26}$$

or, in discretizations form

$$\mathbb{F}(\mathbf{l}, \mathbf{h}, \mathbf{a}) = \sum_{n=1}^{N} \left[\frac{a_n}{l_n} v_{y_2}\left(0, Y_0 h_n^{-1}, t_n\right) - \mu_5(t_n) \right]^2 + \sum_{n=1}^{N} \left[l_n h_n \int_0^1 \int_0^1 v(y_1, y_2, t_n)\, dy_2 dy_1 - \mu_6(t_n) \right]^2 + \sum_{n=1}^{N} \left[l_n h_n^2 \int_0^1 \int_0^1 y_2 v(y_1, y_2, t_n) dy_2 dy_1 - \mu_7(t_n) \right]^2, \tag{27}$$

where $v(y_1, y_2, t)$ solves Eqs. (8) to (11) for given $(l(t), h(t), a(t))$, respectively. The minimization of the function Eq. (27) is performed using the MATLAB subroutine *lsqnonlin* [25].

The inverse problem given by Eqs. (8) to (14) is solved subject to both analytical and noisy (perturbed) measurements Eqs. (12) to (14). The perturbed data are numerically formulated as follows

$$\mu_i^{\epsilon}(t_n) = \mu_i(t_n) + \epsilon_n, \quad n = \overline{1, N}, \quad i = \overline{5, 7}, \tag{28}$$

where ϵ_n are random variables with mean zero and standard deviations σ_i, given by

$$\sigma_i = p \times \max_{t\in[0, T]} |\mu_i(t)|, \quad i = \overline{5, 7}, \tag{29}$$

where p represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\underline{\epsilon} = (\epsilon_n)_{n=\overline{1,N}}$ as follows:

$$\underline{\epsilon} = normrnd(0, \sigma_i, N), \quad i = 1, 2, 3. \tag{30}$$

In the case of perturbed data Eq. (28), we replace in Eq. (27) $\mu_i(t_n)$ by $\mu_i^{\epsilon}(t_n)$ for $i = \overline{5,7}$.

5 Numerical Results and Discussion

The accuracy is measured by *rmse*:

$$rmse(l) = \left[\frac{T}{N}\sum_{n=1}^{N}(l^{numerical}(t_n) - l^{exact}(t_n))^2\right]^{1/2}, \tag{31}$$

$$rmse(h) = \left[\frac{T}{N}\sum_{n=1}^{N}(h^{numerical}(t_n) - h^{exact}(t_n))^2\right]^{1/2}, \tag{32}$$

$$rmse(a) = \left[\frac{T}{N}\sum_{n=1}^{N}(a^{numerical}(t_n) - a^{exact}(t_n))^2\right]^{1/2}. \tag{33}$$

Let us take $T = 1$, for simplicity. The lower bounds and upper bounds for $l(t) > 0$, $h(t) > 0$ and $a(t) > 0$ are 10^{-9} for (LB) and 10^2 for (UB), respectively.

5.1 Example 1

Consider the inverse problem given by Eqs. (1) to (7), with the smooth unknown timewise terms $l(t)$, $h(t)$ and $a(t)$, and we solve it with:

$$\varphi(x_1, x_2) = 1 + \cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right) + \sin(x_1), \quad \mu_1(x_2, t) = 1 + t + \cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right),$$

$$\mu_2(x_2, t) = 1 + t + \cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right) + \sin(1+t), \quad Y_0 = \frac{1}{2}(1+t),$$

$$\mu_3(x_1, t) = 1 + t + \cos\left(\frac{\pi}{8}\right) + \sin(x_1), \quad \mu_4(x_1, t) = 1 + t + \cos\left(\frac{\pi}{8} + \frac{1}{2}\pi(1+t)\right) + \sin(x_1),$$

$$f(x_1, x_2, t) = 1 + \frac{1}{4}\pi^2(2-t)\cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right) + (2-t)\sin(x_1), \tag{34}$$

$$\mu_5(t) = 2 - t, \quad \mu_6(t) = \frac{1}{\pi}(1+t)[\pi(2+t(2+t)) - \pi\cos(1+t) + 2\cos(\frac{1}{8}(\pi + 4\pi t)) - 2\sin\left(\frac{\pi}{8}\right)],$$

$$\mu_7(t) = \frac{1}{2\pi^2}(1+t)[\pi^2(1+t)(2+t(2+t)) - 8\cos(\frac{\pi}{8}) + \pi(1+t)(-\pi\cos(1+t) + 4\cos\left(\frac{1}{8}(\pi + 4\pi t)\right)$$
$$- 8\sin\left(\frac{1}{8}(\pi + 4\pi t)\right)]. \tag{35}$$

We observe that the conditions $(A1)$–$(A6)$ of Theorems 1 and 2 are fulfilled and thus, the uniqueness condition of the solution is guaranteed. It can be easily verified that the analytical solution of Eqs. (1) to (4) is

$$u(x_1, x_2, t) = 1 + t + \sin(x_1) + \cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right), \quad (x_1, x_2, t) \in \Omega_T, \tag{36}$$

and

$$l(t) = 1 + t, \quad h(t) = 1 + t, \quad a(t) = 2 - t, \quad t \in [0, 1]. \tag{37}$$

Also,

$$v(y_1, y_2, t) = u(y_1 l(t), \quad y_2 h(t), t) = 1 + t + \sin((1+t) y_1) + \cos\left(\frac{\pi}{8} + \frac{1}{2}\pi (1+t) y_2\right),$$

$$(y_1, y_2, t) \in Q_T. \tag{38}$$

First, let us solve the forward problem Eqs. (1) to (4) with the input data (34) when $l(t)$, $h(t)$, $a(t)$ are known and given by Eq. (37). Tab. 1 reveals that the exact and approximate solutions for Eqs. (12) to (14), which exactly is given by Eq. (35), obtained with number of grids $M_1 = M_2 = 10$ and $N \in \{20, 40, 80\}$, are in good agreement. The analytical Eq. (38) and approximate solutions for $v(y_1, y_2, t)$ is plotted in Fig. 1. It is clear from Tab. 1 and Fig. 1 that the accuracy of the approximate solution increases, as the mesh sizes decreases.

Table 1: The numerical and analytical Eq. (35) solutions for $\mu_i(t)$, $i = \overline{5, 7}$, with $M_1 = M_2 = 10$ and various $N \in \{20, 40, 80\}$, for direct problem

t	0.1	0.2	0.3	...	0.8	0.9	N	rmse (μ_i)
$\mu_5(t)$	1.9011	1.8021	1.7032	...	1.2043	1.1059	20	1.3E−3
	1.9001	1.8002	1.7003	...	1.2002	1.1001	40	9.3E−4
	1.9000	1.8000	1.7000	...	1.2000	1.1000	80	3.6E−5
	1.9000	1.8000	1.7000	...	1.2000	1.1000	exact	0
$\mu_6(t)$	2.2571	2.7787	3.3689	...	7.5195	8.6382	20	5.8E−3
	2.2589	2.7800	3.3783	...	7.5147	8.6318	40	2.3E−3
	2.2596	2.7810	3.3781	...	7.5128	8.6294	80	8.1E−4
	2.2611	2.7817	3.3700	...	7.5125	8.6280	exact	0
$\mu_7(t)$	1.0712	1.4233	1.8530	...	5.6705	6.9071	20	4.4E−3
	1.0710	1.4231	1.8525	...	5.6651	6.9001	40	7.1E−3
	1.0711	1.4232	1.8527	...	5.6728	6.9023	80	9.3E−4
	1.0746	1.4278	1.8583	...	5.6738	6.9079	Exact	0

In the problem Eqs. (8) to (14), we take the initial guesses for **l**, **h** and **a**, as follows:

$$l^0(t_n) = l_0 = 1, \quad h^0(t_n) = h_0 = 1, \quad a^0(t_n) = a(0) = 2, \quad n = \overline{1, N}. \tag{39}$$

We take a mesh size with $M_1 = M_2 = 10$, $N = 40$ and we start the examination for recovering the timewise terms $l(t)$, $h(t)$ and $a(t)$, when $p = 0$ in the measurements Eqs. (12) to (14), as in Eq. (31). The function Eq. (27) is illustrated in Fig. 2a, where a convergence, which is monotonically decreasing, is obtained in 10 iterations to achieve a low specified tolerance of $O(10^{-28})$.

Figs. 2b–2d illustrates the corresponding approximate results for the functions $l(t)$, $h(t)$ and $a(t)$. A good agreement between the analytical Eq. (37) and approximate solutions can be observed from these figures with $rmse(l) = 4.4\text{E}{-3}$, $rmse(h) = 3.8\text{E}{-3}$ and $rmse(a) = 6.4\text{E}{-3}$.

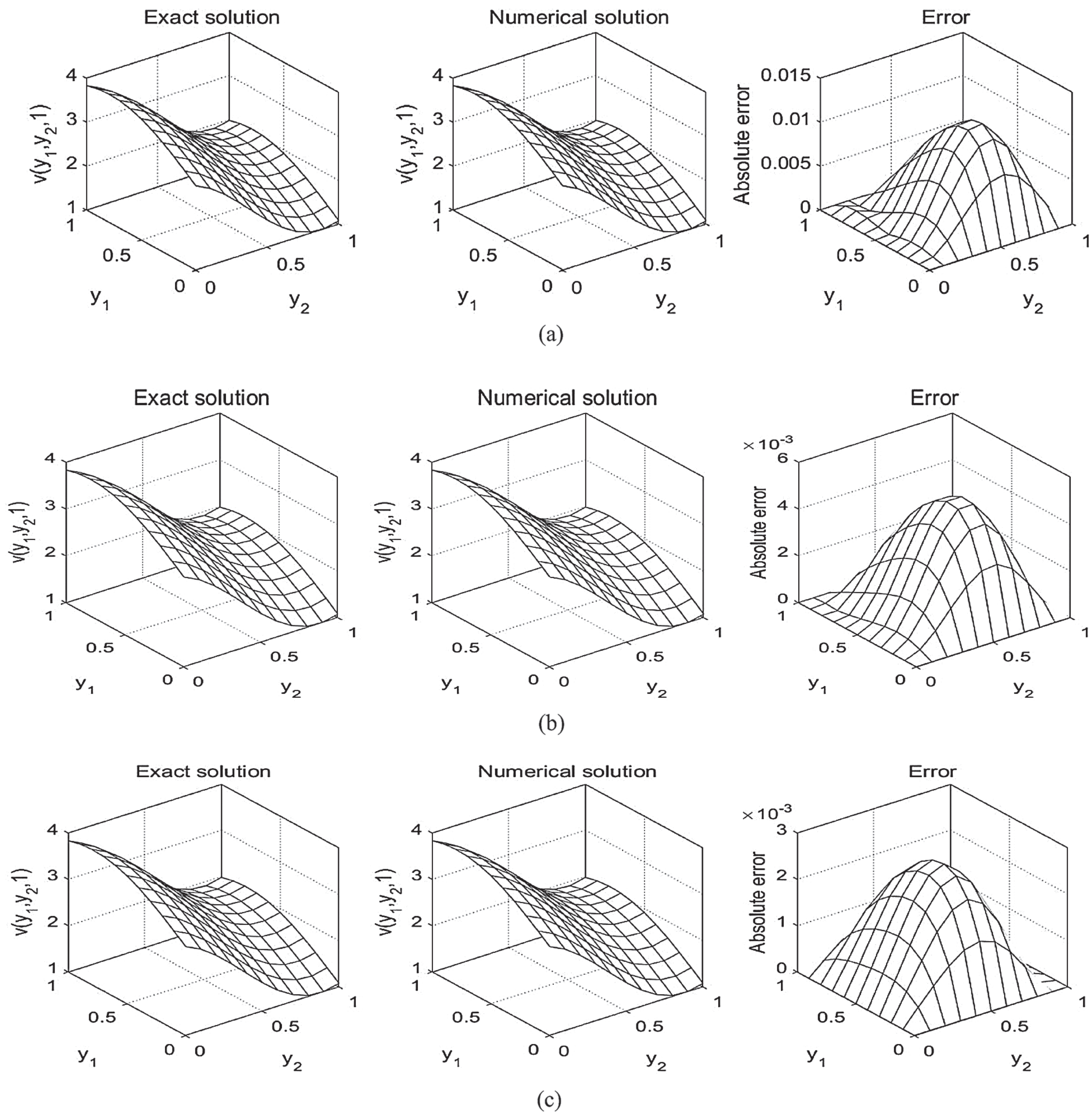


Figure 1: The solutions $v(y_1, y_2, 1)$ and absolute errors with $M_1 = M_2 = 10$ for: (a) $N = 20$, (b) $N = 40$ and (c) $N = 80$, for direct problem

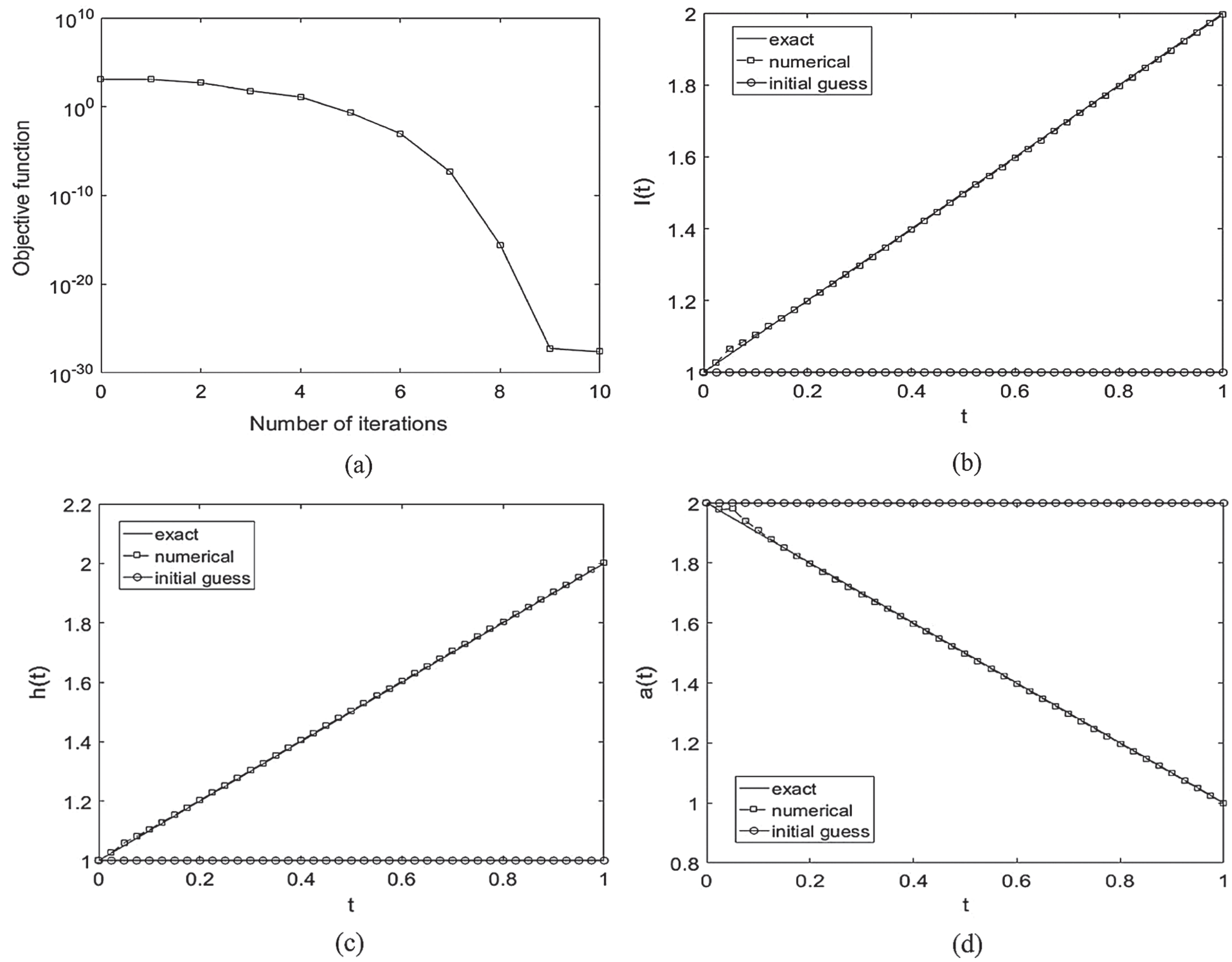


Figure 2: (a) The objective function Eq. (27) *vs*. the number of iterations, and the exact Eq. (37) and numerical solutions for: (b) $l(t)$, (c) $h(t)$ and (d) $a(t)$, with no noise, for Example 1

Next, the stability of the numerical solution is investigated with respect to the noisy data Eq. (28). We include $p \in \{0.1\%, 0.5\%, 1\%, 2\%\}$ noise to the over-determination conditions μ_5, μ_6 and μ_7, as in Eq. (28) to test the stability. The function Eq. (27) is plotted in Fig. 3a and convergence is again rapidly achieved. The approximate results for $l(t)$, $h(t)$ and $a(t)$ are depicted in Fig. 3. Accurate and stable results are obtained for $l(t)$, $h(t)$ and $a(t)$ by observing Figs. 3b–3d and the *rmse* values, the minimum value of Eq. (27) at final iteration and the number of iterations are given in Tab. 2. As noise p is increased the approximate results for $l(t)$, $h(t)$ and $a(t)$ start to build up oscillations. In order to retrieve stability, we penalise the function Eq. (27) by adding $\lambda\left(\left\|l'(t)\right\|^2 + \left\|h'(t)\right\|^2 + \|a(t)\|^2\right)$ to it since the theory provide $l \in C^1[0, T]$, $h \in C^1[0, T]$ and $a \in C[0, T]$, where $\lambda > 0$ is the Tikhonov's regularization parameter to be chosen. Then, in discretised form of Tikhonov functional recasts as

$$\mathbb{F}_\lambda(\mathbf{l}, \mathbf{h}, \mathbf{a}) = \mathbb{F}(\mathbf{l}, \mathbf{h}, \mathbf{a}) + \lambda\left(\sum_{n=1}^{N}\left(\frac{l_n - l_{n-1}}{\Delta t}\right)^2 + \sum_{n=1}^{N}\left(\frac{h_n - h_{n-1}}{\Delta t}\right)^2 + \sum_{n=1}^{N} a_n^2\right). \tag{40}$$

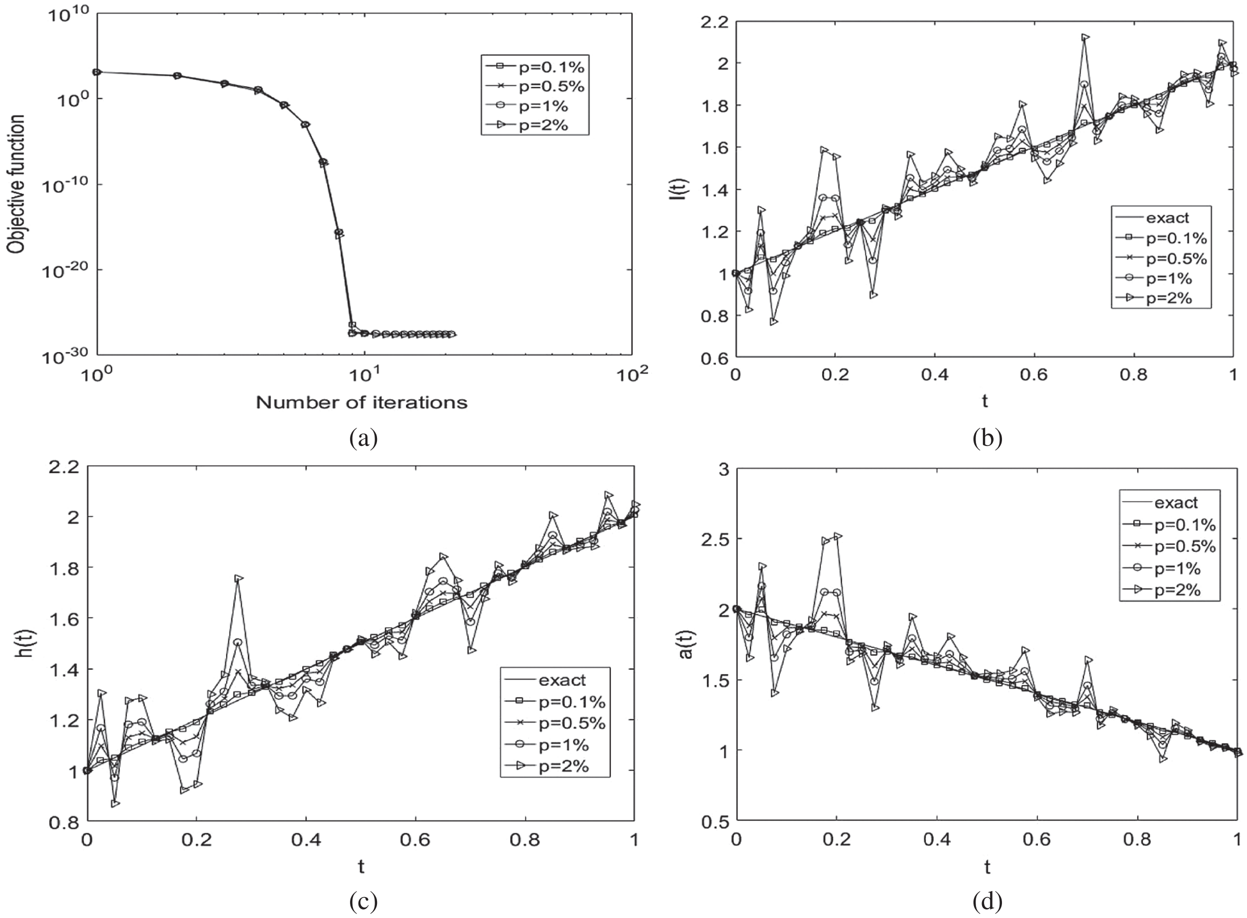


Figure 3: (a) The objective function Eq. (27) *vs*. the number of iterations, and the exact Eq. (37) and numerical solutions for: (b) $l(t)$, (c) $h(t)$ and (d) $a(t)$, with $p \in \{0.1\%, 0.5\%, 1\%, 2\%\}$ noise, for Example 1

Table 2: The values of *rmse* Eqs. (31) to (33), the function Eq. (27) at final iteration and the number of iterations, for $p \in \{0, 0.1\%, 0.5\%, 1\%, 2\%\}$ noise, for Example 1

p (%)	$rmse\,(l)$	$rmse(h)$	$rmse(a)$	Minimum value of (27)	Iter
0	0.0044	0.0038	0.0064	2.5E−28	10
0.1	0.0093	0.0079	0.0122	3.2E−28	10
0.5	0.0427	0.0358	0.0571	3.8E−28	10
1	0.0847	0.0717	0.1141	3.0E−28	21
2	0.1698	0.1448	0.1141	2.3E−28	21

For $p = 5\%$ noise, Fig. 4 illustrates the analytical solution Eq. (37) and the approximate solutions obtained by minimizing the functional Eq. (40) for various regularization parameters. The $rmse(l, h, a)$ values are $\{0.4822, 0.4313, 0.3049, 0.1574, 0.0566, 0.1040\}$, $\{0.4388, 0.3720, 0.2857, 0.1575, 0.0646, 0.1039\}$ and $\{0.6357, 0.5994, 0.4222, 0.2375, 0.1503, 0.2363\}$ for $\lambda \in \{0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$, see Tab. 3 for more information. It can be observed that the approximate unregularized solution obtained with $\lambda = 0$ demonstrates instability, however, on including regularization with $\lambda = 10^{-3}$ to $\lambda = 10^{-2}$, we get a stable solution which is consistent in accuracy with the $p = 5\%$ noise desecrating the input data Eq. (28).

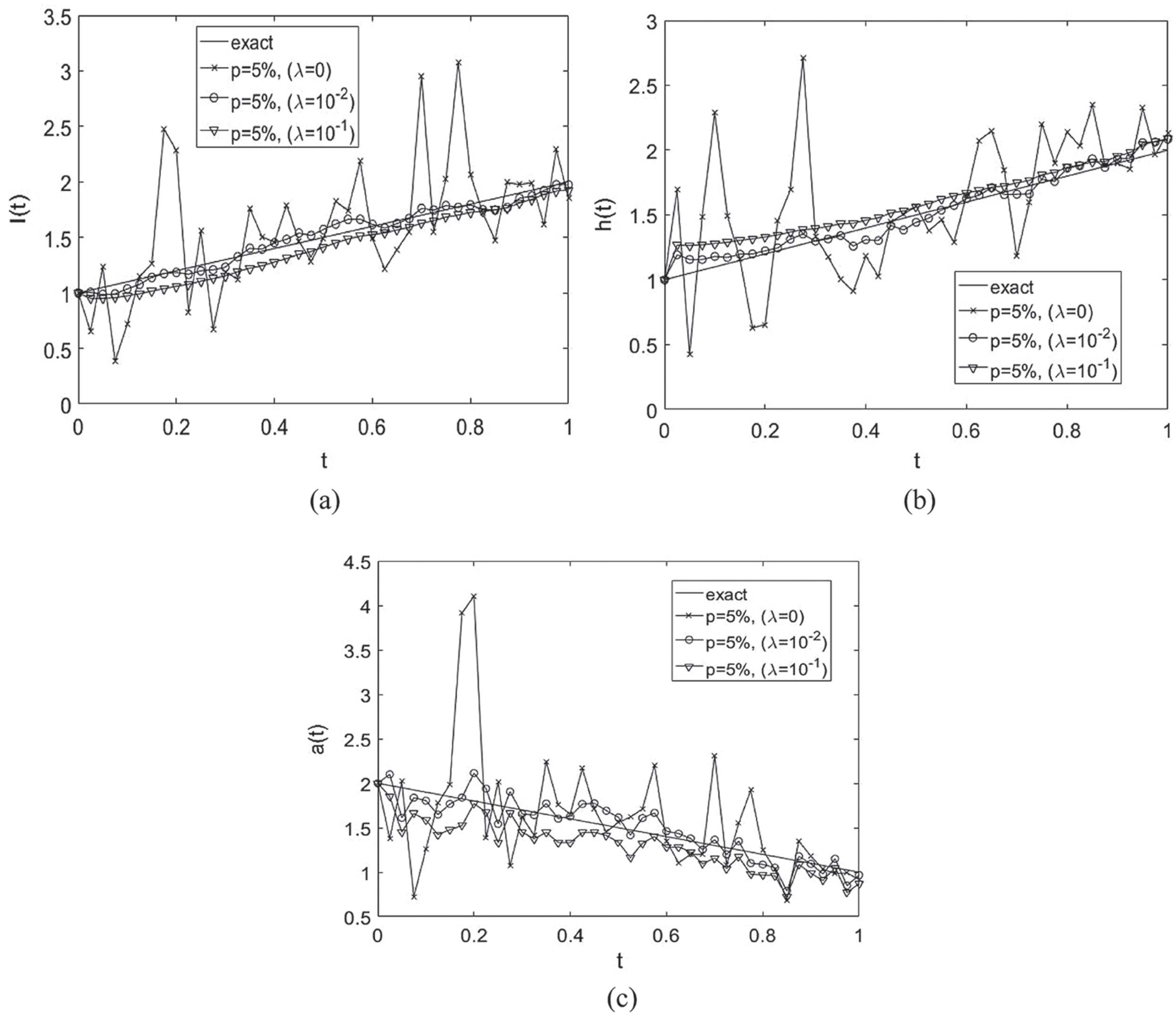


Figure 4: The numerical and analytical Eq. (37) solutions for: (a) $l(t)$, (b) $h(t)$ and (c) $a(t)$, for $p = 5\%$ noise, with various $\lambda \in \{0, 10^{-2}, 10^{-1}\}$, for Example 1

Table 3: The values of *rmse* Eqs. (31) to (33) and the number of iterations, for $p = 5\%$ noise, without and with regularization, for Example 1

p (%)	λ	$rmse\,(l)$	$rmse\,(h)$	$rmse\,(a)$	Iter
5	0	0.4822	0.4388	0.6357	50
5	10^{-5}	0.4313	0.3720	0.5994	10
5	10^{-4}	0.3049	0.2857	0.4222	10
5	10^{-3}	0.1574	0.1575	0.2375	10
5	10^{-2}	0.0566	0.0646	0.1503	10
5	10^{-1}	0.1040	0.1039	0.2363	10

5.2 *Example 2*

The smooth timewise coefficients $l(t)$, $h(t)$, $a(t)$ given by Eq. (37) has been recovered in previous example. In this example, let us examine the numerical scheme for recovering a non-smooth test:

$$l(t) = \frac{1}{\sqrt{1+t}}, \quad h(t) = \frac{1}{\sqrt{1+t}}, \quad a(t) = 1 + \cos^2(2\pi t), \quad t \in [0, 1] \tag{41}$$

and the analytical solution for the temperature $u(x_1, x_2, t)$ given by Eq. (36). The rest of the input data are given as

$$\mu_2\left(x_2, t\right) = 1 + t + \cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right) + \sin\left(\frac{1}{\sqrt{1+t}}\right), \quad Y_0 = \frac{1}{2\sqrt{1+t}},$$

$$\mu_4\left(x_1, t\right) = 1 + t + \cos\left(\frac{\pi}{8} + \frac{\pi}{2\sqrt{1+t}}\right) + \sin\left(x_1\right), \quad \mu_5\left(t\right) = 1 + \cos^2\left(2\pi t\right),$$

$$\begin{aligned}\mu_6\left(t\right) = {} & \frac{1}{\pi\sqrt{1+t}}[\pi + \pi\sqrt{1+t} - \pi\cos(\frac{1}{\sqrt{1+t}}) - 2\sin(\frac{\pi}{8}) \\ & + 2\sin\left(\frac{1}{8}\pi\left(1 + \frac{4}{\sqrt{1+t}}\right)\right)], \mu_7(t) = \frac{1}{2\pi^2\sqrt{(1+t)^3}}[\pi^2(1 + t + \sqrt{1+t}) \\ & - 8\left(1+t\right)\cos\left(\frac{\pi}{8}\right) + 8\left(1+t\right)\cos\left(\frac{1}{8}\pi\left(1 + \frac{4}{\sqrt{1+t}}\right)\right) + \pi\sqrt{1+t}(-\pi\cos(\frac{1}{\sqrt{1+t}}) \\ & + 4\sin\left(\frac{1}{8}\pi\left(1 + \frac{4}{\sqrt{1+t}}\right)\right))], \\ f(x_1, x_2, t) = {} & 1 + \frac{1}{4}\pi^2(1 + \cos^2(2\pi t))\cos\left(\frac{\pi}{8} + \frac{\pi x_2}{2}\right) + (1 + \cos^2(2\pi t))\sin(x_1).\end{aligned} \tag{42}$$

Also,

$$v\left(y_1, y_2, t\right) = u\left(y_1 l\left(t\right), y_2 h\left(t\right), t\right) = 1 + t + \sin\left(\frac{y_1}{\sqrt{1+t}}\right) + \cos\left(\frac{\pi}{8} + \frac{\pi y_2}{2\sqrt{1+t}}\right), \quad (y_1, y_2, t) \in Q_T. \tag{43}$$

With this data, the conditions $(A1)$–$(A6)$ of Theorems 1 and 2 are holds, thus the uniqueness of the solution is also ensured. The initial guesses for **l**, **h** and **a**, for this example has been taken as in Eq. (39).

As in Example 1, we take $\Delta y_1 = \Delta y_2 = 0.1$ and $\Delta t = 0.025$ and we first choose the case when $p = 0$ in the data μ_5, μ_6 and μ_7, in Eq. (28). The analytical Eq. (41) and numerical solutions for $l(t)$, $h(t)$ and $a(t)$ is illustrated in Fig. 5, where the reconstructed timewise terms are in excellent agreement with their corresponding analytical solutions, obtaining with $rmse(l) = 6.3\text{E}{-3}$, $rmse(h) = 1.7\text{E}{-3}$ and $rmse(a) = 9.8\text{E}{-3}$, see Tab. 4.

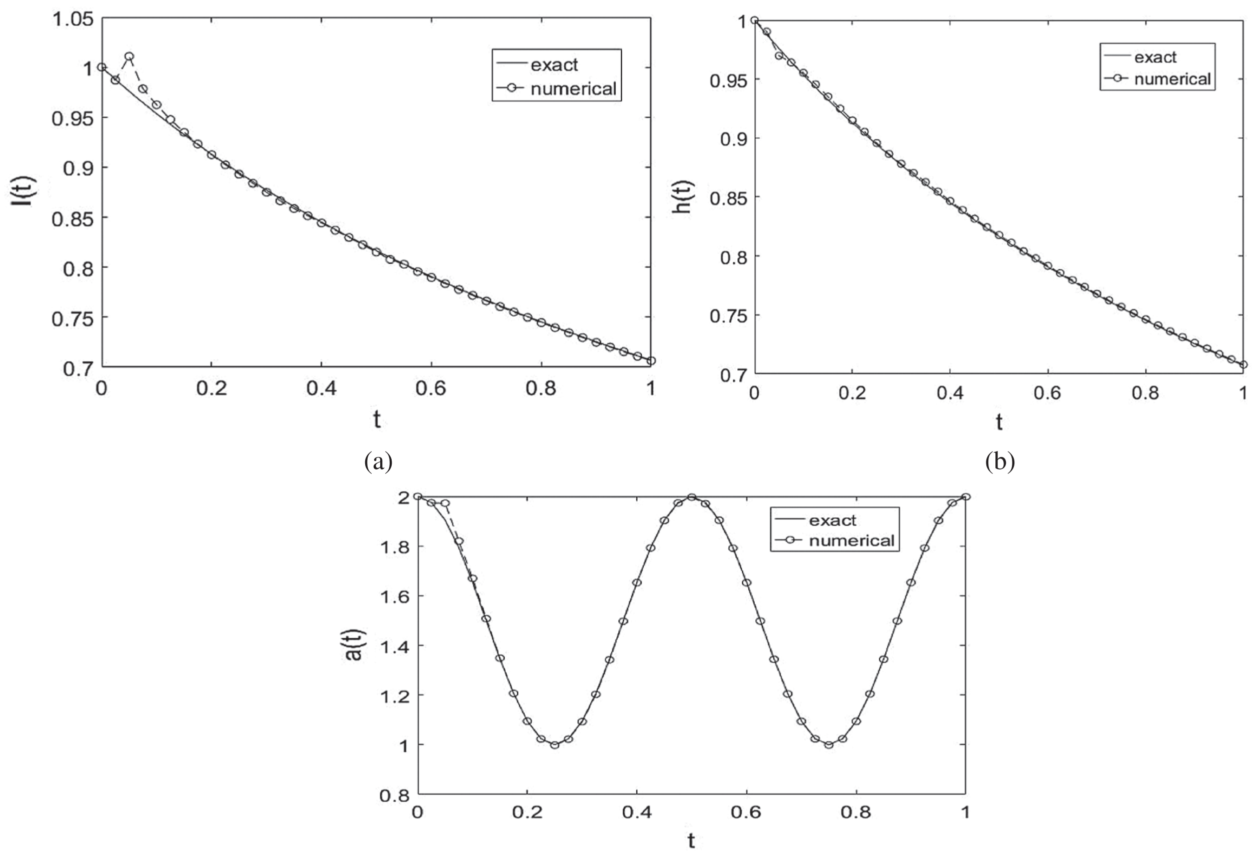


Figure 5: The analytical Eq. (41) and numerical solutions for: (a) $l(t)$, (b) $h(t)$ and (c) $a(t)$, with no noise and no regularization, for Example 2

Next, we add $p \in \{0.5\%, 1\%, 2\%\}$ noise into the measured data Eqs. (12) to (14), in order to investigate the stability of the numerical results. The approximate results for the free boundaries $l(t)$, $h(t)$ and the thermal conductivity $a(t)$ are illustrated in Fig. 6. From this figure it can be observed that reasonable and stable numerical results are obtained. The numerical solutions for $l(t)$, $h(t)$ and $a(t)$ obtained with the $p = 5\%$ noise have been found highly oscillatory and unstable when no regularization was employed, i.e., $\lambda = 0$, obtaining $rmse(l) = 0.1317$, $rmse(h) = 0.2439$ and $rmse(a) = 0.2779$, see Fig. 7. Therefore, regularization is required in order to restore the stability of the solution in the components $l(t)$, $h(t)$ and $a(t)$. Including regularization, i.e., $\lambda \in \{10^{-3}, 10^{-2}\}$, in Eq. (40) alleviates this instability, as shown in the regularized approximate results in Fig. 7 and Tab. 5. The absolute errors between the analytical Eq. (43) and approximate $v(y_1, y_2, t)$ with the $p = 5\%$ noise, and without and with regularization parameters, is illustrated in Fig. 8. From this figure and Tab. 5 it can be noticed that the temperature $v(y_1, y_2, t)$ component is stable and accurate. Overall, the same conclusions can be carried out about the stable determination for the unknown time-dependent coefficients.

Table 4: The values of *rmse* Eqs. (31) to (33), the function Eq. (27) at final iteration and the number of iterations, for $p \in \{0, 0.5\%, 1\%, 2\%\}$ noise, for Example 2

p (%)	$rmse\,(l)$	$rmse\,(h)$	$rmse\,(a)$	Minimum value of (27)	Iter
0	0.0063	0.0017	0.0098	3.5E−29	15
0.5	0.0146	0.0083	0.0276	1.8E−28	8
1	0.0385	0.0206	0.0704	9.4E−29	10
2	0.0625	0.0423	0.1180	5.8E−29	10

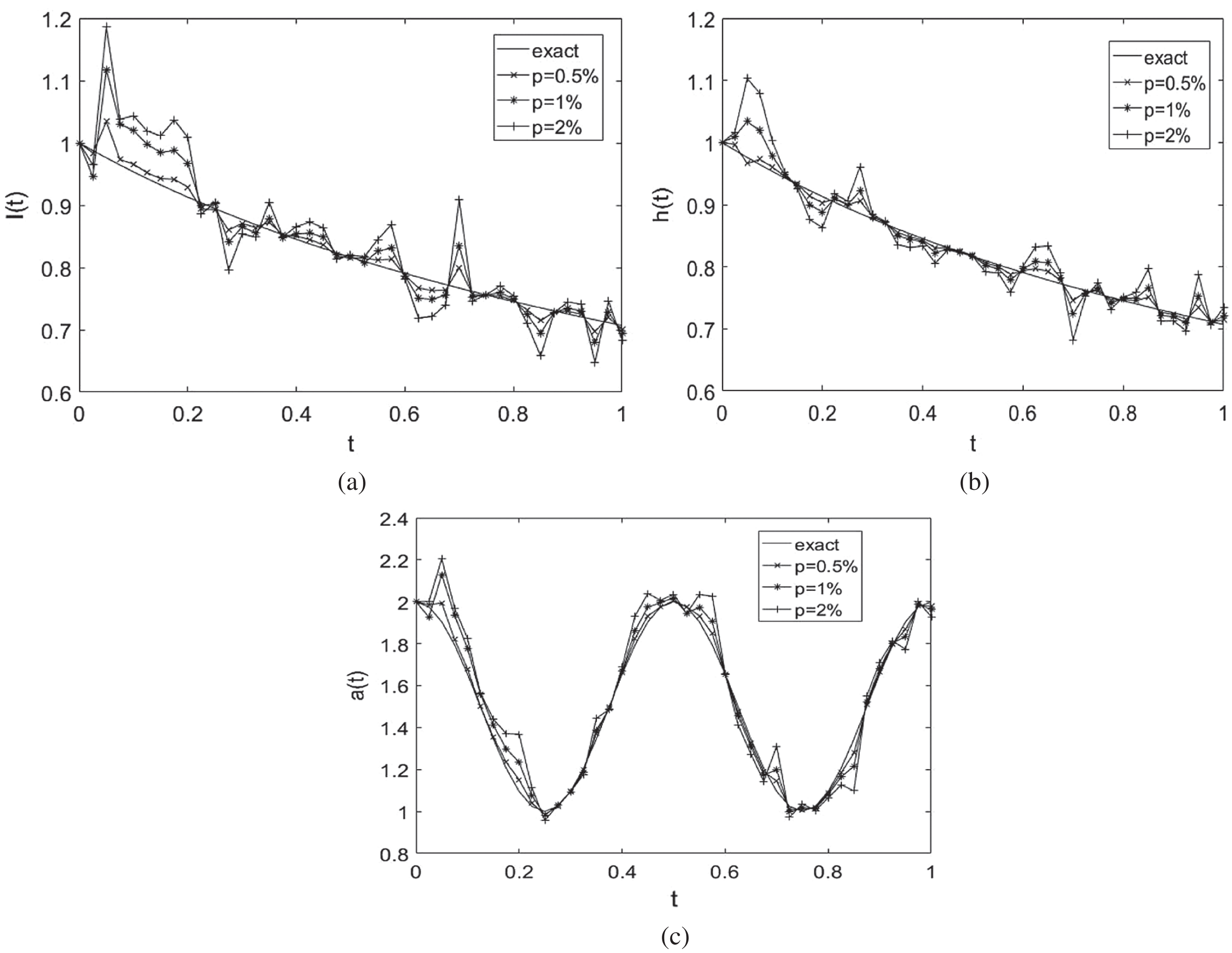
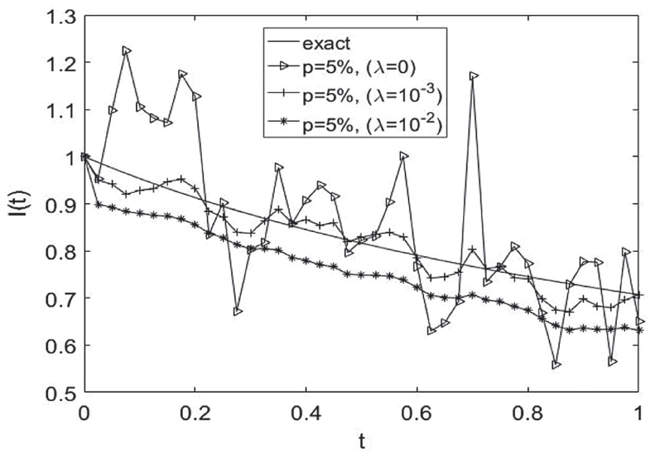
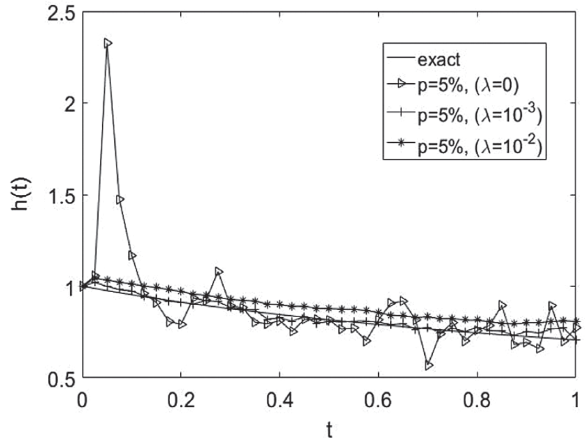


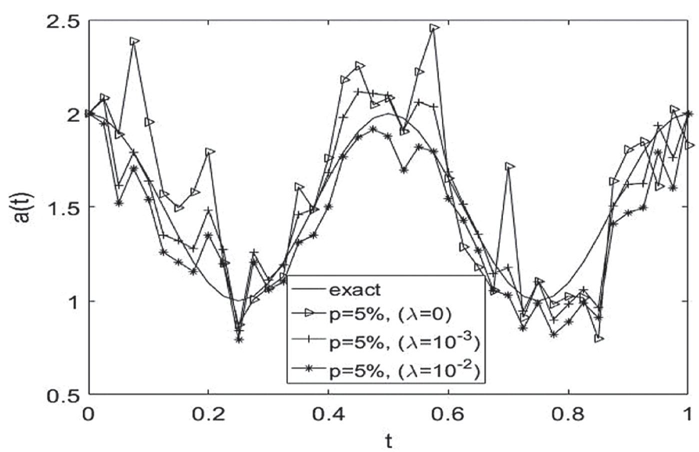
Figure 6: The analytical Eq. (41) and numerical solutions for: (a) $l(t)$, (b) $h(t)$ and (c) $a(t)$, with $p \in \{0.5\%, 1\%, 2\%\}$ noise, for Example 2



(a)



(b)



(c)

Figure 7: The analytical Eq. (41) and numerical solutions for: (a) $l(t)$, (b) $h(t)$ and (c) $a(t)$, for $p = 5\%$ noise and without and with regularization, for Example 2

Table 5: The values of *rmse* Eqs. (31) to (33) and the number of iterations, for $p = 5\%$ noise, without and with regularization, for Example 2

p (%)	λ	*rmse* (l)	*rmse* (h)	*rmse* (a)	Iter
5	0	0.1317	0.2439	0.2779	18
5	10^{-5}	0.0979	0.0730	0.2217	10
5	10^{-4}	0.0628	0.0460	0.1815	10
5	10^{-3}	0.0293	0.0208	0.1514	10
5	10^{-2}	0.0712	0.0623	0.1776	10
5	10^{-1}	0.2728	0.2964	0.5437	10

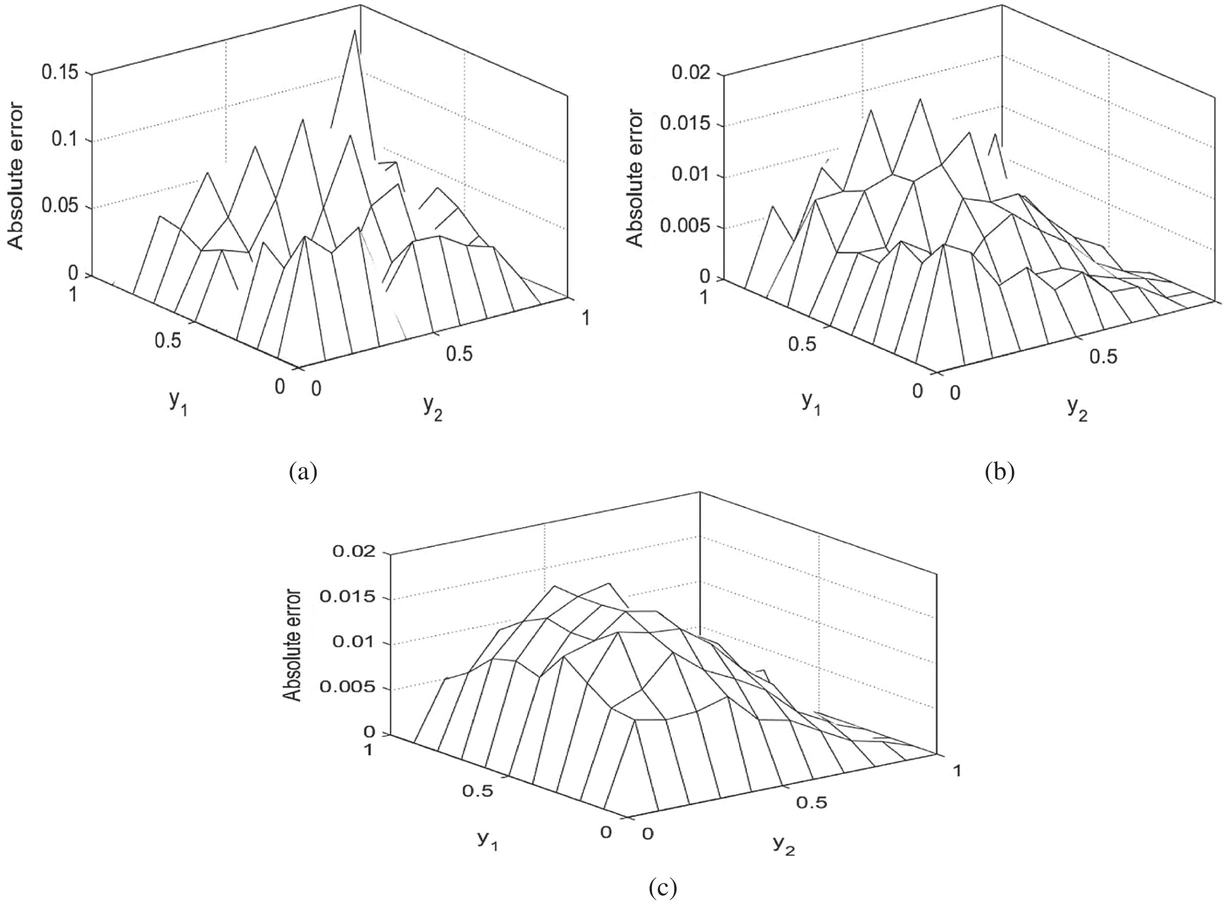


Figure 8: The absolute errors between the analytical Eq. (43) and numerical $v(y_1, y_2, t)$ with (a) $\lambda = 0$, (b) $\lambda = 10^{-3}$ and (c) $\lambda = 10^{-2}$, for $p = 5\%$, for Example 2

6 Conclusions

The inverse problem concerning the reconstruction of the time-dependent thermal conductivity $a(t)$, and free boundary coefficients $l(t)$ and $h(t)$ along with the temperature $u(x_1, x_2, t)$ in a two-dimensional parabolic equation from the over-specification conditions has been solved for the first time numerically. The forward solver based on the ADE was employed. The inverse problem approach based on a nonlinear least-squares minimization problem using the MATLAB optimization subroutine was developed. The Tikhonov regularization has been employed in order to obtain stable and accurate solutions since the inverse problem is ill-posed. The numerical results for the inverse problem shows that stable and accurate approximate results have been obtained. Extension to three-dimensions is in principle straightforward.

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