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Article

Power Inverted Topp–Leone Distribution in Acceptance Sampling Plans

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Abstract: We introduce a new two-parameter model related to the inverted Topp–Leone distribution called the power inverted Topp–Leone (PITL) distribution. Major properties of the PITL distribution are stated; including; quantile measures, moments, moment generating function, probability weighted moments, Bonferroni and Lorenz curve, stochastic ordering, incomplete moments, residual life function, and entropy measure. Acceptance sampling plans are developed for the PITL distribution, when the life test is truncated at a pre-specified time. The truncation time is assumed to be the median lifetime of the PITL distribution with pre-specified factors. The minimum sample size necessary to ensure the specified life test is obtained under a given consumer's risk. Numerical results for given consumer's risk, parameters of the PITL distribution and the truncation time are obtained. The estimation of the model parameters is argued using maximum likelihood, least squares, weighted least squares, maximum product of spacing and Bayesian methods. A simulation study is confirmed to evaluate and compare the behavior of different estimates. Two real data applications are afforded in order to examine the flexibility of the proposed model compared with some others distributions. The results show that the power inverted Topp–Leone distribution is the best according to the model selection criteria than other competitive models.

Keywords: Inverted Topp–Leone distribution; acceptance sampling plans; maximum likelihood estimators; weighted least squares estimators; Bayesian estimators

1 Introduction

The inverted (inverse) distributions have considerable applications in several area including; biological sciences, life testing problems, survey sampling, engineering sciences, etc. Many inverted distributions and their applications have been devoted by several authors; for instance, Keller et al. [1] studied the shapes of the density and failure rate functions for the inverse Weibull model. Reference [2] proposed a generalized inverse Weibull distribution with decreasing and unimodal failure rates. Reference [3] proposed the inverse Lindley distribution and studied its main properties. The inverted Kumaraswamy distribution has been discussed in [4]. The inverted Nadarajah–Haghighi distribution with decreasing and upside-down bathtub hazard rate was discussed in [5]. Reference [6] proposed and studied the inverse power Lomax distribution. Reference [7] introduced inverted exponentiated Lomax distribution and estimated the distribution for right censored data. Reference [8] introduced the inverted Topp–Leone (**ITL**) distribution and discussed several properties.

The cumulative distribution function (**CDF**) of random variable Y has the ITL distribution with shape parameter $\alpha > 0$ is defined by:

$$F_{ITL}(y;\alpha) = 1 - \left\{ \frac{(1+2y)^{\alpha}}{(1+y)^{2\alpha}} \right\}; \quad y \geq 0,\ \alpha > 0. \tag{1}$$

The probability density function (**PDF**) related to (1) is given by

$$f_{ITL}(y;\alpha) = 2\alpha y\,(1+y)^{-2\alpha-1}\,(1+2y)^{\alpha-1}; \quad y,\ \alpha > 0. \tag{2}$$

In recent times, several extended and generalized formulations of the classical distributions, based on different procedures, have been discussed by several authors (see for example [9–13]). The power transformation (**PT**) approach is one of the most important methods that have been employed for this purpose. It is employed to create new distributions out of the well-known distributions through adding an additional parameter. This approach allows more flexible model able to describe different types of real data. PT procedure for several distributions has been provided by several researches (see, for example [14–16]).

Acceptance sampling (**AS**) concerns with inspection and decision-making regarding lots of product and constitutes one of the oldest techniques in quality assurance. A typical application of AS is as follows:

Required: A company receives a shipment of product from a vendor. This product is often a component or raw material used in the company's manufacturing process.

- ***Sampling:*** A sample is taken from the lot and the relevant quality characteristic of the units in the sample is inspected.
- ***Decision:*** On the basis of the information of the given sample, a decision is made regarding lot disposition to accept or to reject the lot.
  1. **For AS:** Accepted lots are put into production,
  2. **For rejected samples:** Rejected lots may be returned to the vendor or may be subjected to some other lot disposition action.

The objective of this research is to provide a generalized formula of the ITL model by employing the PT as $X = Y^{1/\theta}$, where Y has the ITL distribution. We call the modified form of ITL model as the PITL distribution. The PITL model is able to **(i)** give favorite properties owing to the additional shape parameter; **(ii)** give more flexibility of the PDF and hazard rate function (**HRF**); **(iii)** provide more flexibility of the kurtosis compared to ITL model; **(iv)** develop a sampling plan, derive its operating characteristic function and give the corresponding decision; **(v)** estimate the model parameters based on different methods of estimation, and **(vi)** analyze two read data.

This paper involves the following sections. In Section 2, we introduce the two-parameter PITL distribution. Section 3 gives some stractural properties of the PITL distribution. The design of proposed AS plan under a truncated life test is discussed in Section 4. Section 5 discusses parameter estimation of the PITL model based on the maximum likelihood (**ML**), maximum product of spacing (**MPS**), least squares (**LS**), weighted LS (**WLS**) and Bayesian methods. Section 6 provides a numerical study. Real data are analyzed in Section 7 and the article finishes with concluding remarks.

2 Power Inverted Topp–Leone Distribution

In this section, we define a new probability distribution related to the ITL distribution via a PT method. The formulae of its PDF, CDF, survival function (**SF**), HRF and cumulative HRF are given.

Definition:

A random variable X is said to have the PITL distribution if we employ the PT $X = Y^{1/\theta}$, *where Y has the ITL distribution with CDF(1). The CDF of a random variable has the PITL distribution with shape parameters* α and θ, *denoted by* $X \sim PITL\ (\alpha, \theta)$, *is defined by*

$$F_{PITL}\left(x;\alpha,\theta\right) = 1 - \left\{ \frac{\left(1+2x^{\theta}\right)^{\alpha}}{\left(1+x^{\theta}\right)^{2\alpha}} \right\}; \quad x \geq 0,\ \theta,\ \alpha > 0. \tag{3}$$

The PDF of the PITL distribution related to (3) is given by:

$$f_{PITL}\left(x;\alpha,\theta\right) = 2\alpha\theta x^{2\theta-1}\left(1+x^{\theta}\right)^{-2\alpha-1}\left(1+2x^{\theta}\right)^{\alpha-1}; \quad x,\ \theta,\ \alpha > 0. \tag{4}$$

For, $\theta = 1$, the PDF (4) provides the ITL distribution (see [8]). The survival function; $\overline{F}\left(x;\alpha,\theta\right)$, and the HRF; $h\left(x;\alpha,\theta\right)$ of the PITL distribution are, respectively, given by

$$\overline{F}_{PITL}\left(x;\alpha,\theta\right) = \left\{ \frac{\left(1+2x^{\theta}\right)^{\alpha}}{\left(1+x^{\theta}\right)^{2\alpha}} \right\}, \tag{5}$$

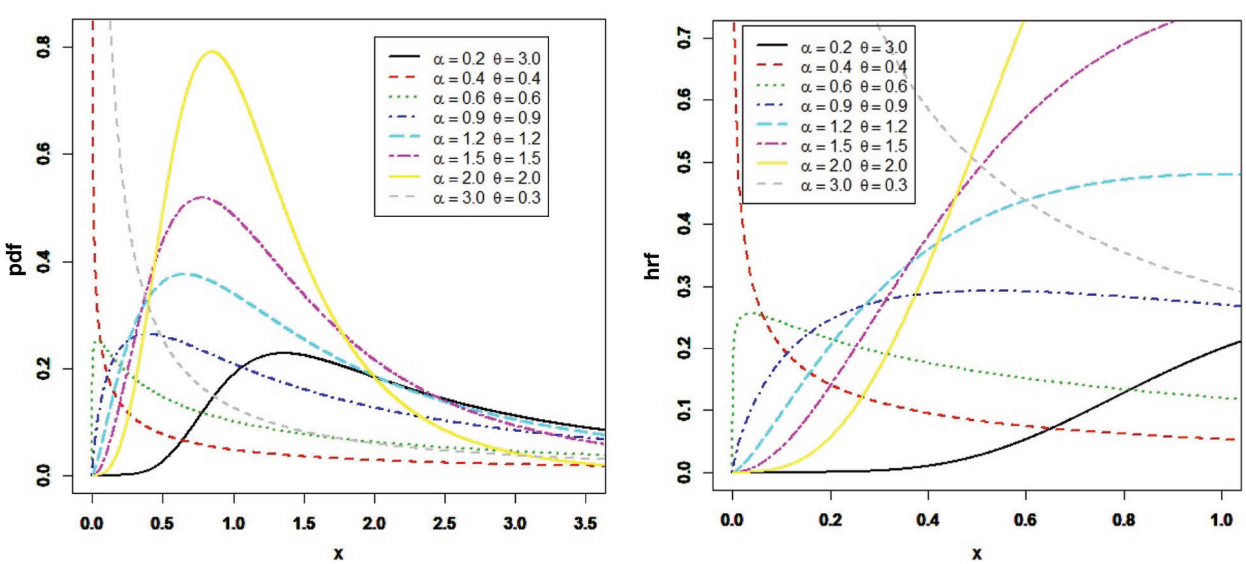

Figure 1: The PDF and HRF plots for the PITL distribution

and

$$h(x;\alpha,\theta) = 2\alpha\,\theta\,x^{2\theta-1}\left[\left(1+x^{\theta}\right)\left(1+2x^{\theta}\right)\right]^{-1}. \tag{6}$$

Plots of the PDF and HRF are presented in Fig. 1 for some choice's values of parameters.

The shape of the PITL PDF could be inverted bathtub, reversed J-shape, unimodal, and positively skewed. The shape of the HRF of the PITL shows that it is increasing, decreasing, reversed J-shape and up-side down.

3 Structural Properties

This section gives some necessary characteristics of the PITL distribution such as; the probability weighted moments, the kth moment, the moment-generating function (**MGF**), inequality measures, rth moment of the residual lifetime (**RL**), Rényi entropy, and stochastic ordering.

3.1 Probability Weighted Moments

The probability weighted moments (**PWM**) are ordinarily used to find estimators of the parameters and quantiles of distributions. The PWM of X (for $r \geq 1$, $s \geq 0$) is defined by:

$$\Xi_{r,s} = \int_0^{\infty} x^r f(x)\,[F(x)]^s\,\mathrm{d}x. \tag{7}$$

Use binomial expansion for $[F_{PITL}(x;\ \alpha,\ \theta)]^s$ as follows:

$$[F_{PITL}(x;\alpha,\theta)]^s = \sum_{m=0}^{s}(-1)^m\binom{s}{m}\left(1+x^{\theta}\right)^{-\alpha m}\left\{1+\frac{x^{\theta}}{1+x^{\theta}}\right\}^{\alpha m}. \tag{8}$$

The PWM of the PITL distribution is obtained by substituting PDF (4) and CDF (8) in (7) as follows:

$$\Xi_{r,s} = 2\theta\alpha\sum_{m=0}^{s}(-1)^m\binom{s}{m}\int_0^{\infty} x^{r+2\theta-1}\left(1+x^{\theta}\right)^{-\alpha m-\alpha-2}\left\{1+\frac{x^{\theta}}{1+x^{\theta}}\right\}^{\alpha m+\alpha-1} dx. \tag{9}$$

Employ the following generalized binomial expansion, where $b>0$ is real non integer and $|z|<1$,

$$(1+z)^{b-1} = \sum_{i=0}^{\infty}\binom{b-1}{j} z^i, \tag{10}$$

in $\Xi_{r,s}$ then we get

$$\Xi_{r,s} = \frac{W_{i,m}}{\Gamma(\alpha m+\alpha+i+2)}\Gamma\left(\frac{r}{\theta}+i+2\right)\Gamma\left(\alpha m+\alpha-\frac{r}{\theta}\right), \tag{11}$$

where $W_{i,m} = 2\alpha\sum_{i=0}^{\infty}\sum_{m=0}^{s}(-1)^m\binom{s}{m}\binom{\alpha+\alpha m-1}{i}$ and $\Gamma(.)$ is the gamma function.

3.2 Moments and Quantiles

Here, we present the kth moment, MGF, and quantile analysis of the PITL $(x;\ \alpha,\ \theta)$ distribution. The kth moment for the PITL is derived as follows:

$$\mu_k' = 2\alpha \sum_{\ell=0}^{\infty} \binom{\alpha-1}{\ell} \frac{\Gamma\left(\frac{k}{\theta}+\ell+2\right)\Gamma\left(\alpha-\frac{k}{\theta}\right)}{\Gamma\left(\ell+2+\alpha\right)}, \quad \alpha\theta > k. \tag{12}$$

The first four moments about zero are obtained after putting $k = 1, 2, 3, 4$ in (12). The MGF of the PITL distribution is given by

$$M_x(t) = E\left(e^{tx}\right) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mu_k' = \sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{2\alpha t^k}{k!} \binom{\alpha-1}{\ell} \mathrm{B}\left(\frac{k}{\theta}+\ell+2, \alpha-\frac{k}{\theta}\right). \tag{13}$$

The kth central moment (μ_k) of the PITL distribution is given by:

$$\mu_k = E\left(X-\mu_1'\right)^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\mu_1'\right)^i \mu_{k-i}' \tag{14}$$

Moreover, we obtain quantile function of the PITL, say $x_p = Q(p) = F^{-1}(u)$, by inverting (3) as follows:

$$x_p = \left[\frac{-2\left((1-p)^{1/\alpha}-1\right) + \sqrt{4\left((1-p)^{1/\alpha}-1\right)^2 - 4(1-p)^{1/\alpha}\left((1-p)^{1/\alpha}-1\right)}}{2(1-p)^{1/\alpha}}\right]^{\frac{1}{\theta}}. \tag{15}$$

In particular, the first three quartiles, say Q_1, Q_2 and Q_3 are obtained by setting $u = 0.25$, 0.5, 0.75 respectively, in (15).

3.3 Inequality Measures

The Bonferroni curve (**BC**) as well as Lorenz curve (**LC**) are widely useful not only in economics to study income and poverty, but also in other fields, such as reliability, insurance and medicine. The LC and BC of the PITL model are derived, respectively, as follows:

$$LC(t) = \frac{E(X\,|x<t)}{E(X)} = \frac{\sum_{j=0}^{\infty}\binom{\alpha-1}{j}\mathrm{B}\left(\frac{1}{\theta}+j+2, \alpha-\frac{1}{\theta}, \frac{t^\theta}{1+t^\theta}\right)}{\sum_{\ell=0}^{\infty}\binom{\alpha-1}{\ell}\frac{\Gamma\left(\frac{1}{\theta}+\ell+2\right)\Gamma\left(\alpha-\frac{1}{\theta}\right)}{\Gamma(\ell+2+\alpha)}}, \tag{16}$$

and

$$BC(t) = \frac{LC(t)}{F_{PITL}(t)} = \frac{\sum_{j=0}^{\infty}\binom{\alpha-1}{j}\mathrm{B}\left(\frac{1}{\theta}+j+2, \alpha-\frac{1}{\theta}, \frac{t^\theta}{1+t^\theta}\right)}{\left\{1-\left[(1+2t)^\alpha(1+t)^{-2\alpha}\right]\right\}\sum_{\ell=0}^{\infty}\binom{\alpha-1}{\ell}\frac{\Gamma\left(\frac{1}{\theta}+\ell+2\right)\Gamma\left(\alpha-\frac{1}{\theta}\right)}{\Gamma(\ell+2+\alpha)}}, \tag{17}$$

where $B\left(.,.,t^{\theta}/1+t^{\theta}\right)$ is the incomplete beta function.

3.4 Residual and Reversed Residual Life Functions

Here we obtain the rth moment of the RL of the PITL model. The rth moment of RL is defined as follows

$$\varpi_r(t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} (x-t)^r f(x)\, dx. \tag{18}$$

The rth moment of the RL of the PITL distribution is derived by using the binomial expansion and the PDF (4) in (18), as follows:

$$\varpi_r(t) = \frac{2\alpha}{\overline{F}_{PITL}(t;\alpha,\theta)} \sum_{j=0}^{r} \sum_{\ell=0}^{\infty} (-t)^{r-j} \binom{n}{j} \binom{\alpha-1}{\ell} t^{r-j} \mathrm{B}\left(\frac{j}{\theta}+\ell+2, \alpha-\frac{j}{\theta}, 1/1+t^{\theta}\right). \tag{19}$$

An important application of the moments of RL is the mean which represents the expected additional life length for an item which is working at age t and obtained by putting $r=1$ in (19).

On contrast, the reversed RL is defined as the conditional random variable $t-X \mid X \le t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. The rth moment of the reversed RL for PITL distribution is given by

$$\phi_r(t) == \frac{2\alpha}{F_{PITL}(t;\alpha,\theta)} \sum_{j=0}^{r} \sum_{\ell=0}^{\infty} (-t)^{r-j} \binom{r}{j} \binom{\alpha-1}{\ell} t^{r-j} \mathrm{B}\left(\frac{j}{\theta}+\ell+2, \alpha-\frac{j}{\theta}, t^{\theta}/1+t^{\theta}\right). \tag{20}$$

The mean of reversed RL serves as the waiting time elapsed since the failure of an item on condition that this failure had occurred.

3.5 Rényi and ω-Entropies

The entropy of a random variable is a measure of the uncertainty variation. The Rényi entropy of PITL distribution is obtained as follows:

$$\begin{aligned} RE(X) &= (1-\gamma)^{-1} \log\left\{\int_0^{\infty} (2\alpha\theta)^{\gamma} x^{\gamma(2\theta-1)} \left(1+x^{\theta}\right)^{-(2\alpha+1)\gamma} \left(1+2x^{\theta}\right)^{(\alpha-1)\gamma} \mathrm{d}x\right\} \\ &= (1-\gamma)^{-1} \log\left\{(2\alpha)^{\gamma} \sum_{i=0}^{\infty} \theta^{\gamma-1} \binom{(\alpha-1)\gamma}{i} \mathrm{B}\left(2\gamma-\frac{\gamma}{\theta}+i+\frac{1}{\theta}, \gamma\alpha+\frac{\gamma}{\theta}-\frac{1}{\theta}\right)\right\}. \end{aligned} \tag{21}$$

The ω-entropy is defined by

$$H_{\omega}(X) = \frac{1}{\omega-1} \log\left(1-\int_0^{\infty} f(x)^{\omega}\, \mathrm{d}x\right), \quad \omega > 0 \text{ and } \omega \ne 1. \tag{22}$$

Therefore, the ω-entropy of the PITL distribution is given by

$$H_{\omega}(X) = \frac{1}{\omega-1} \log\left\{1-(2\alpha)^{\omega} \sum_{i=0}^{\infty} \theta^{\omega-1} \binom{(\alpha-1)\omega}{i} \mathrm{B}\left(2\omega-\frac{\omega}{\theta}+i+\frac{1}{\theta}, \omega\alpha+\frac{\omega}{\theta}-\frac{1}{\theta}\right)\right\}. \tag{23}$$

3.6 Stochastic Ordering

Let X and Y are independent random variables with CDFs F_x and F_y respectively, X is said to be smaller than Y if the following ordering holds (see [17]):

- Stochastic order ($X \leq_{sr} Y$) if $F_X(x) \geq F_Y(x)$ for x.
- Likelihood ratio order ($X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ is decreasing in x.
- Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x.
- Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x.

We have the following chain of implications among the various partial orderings mentioned above:

$$\begin{array}{lll} X \leq_{lr} Y \Rightarrow X & \leq_{hr} & Y \Rightarrow X \leq_{mrl} Y \\ & \Downarrow & \\ & X \leq_{sr} Y & \end{array}$$

To show that the random variable X is smaller than Y, where X and Y have the PITL with different parameters, so we prove the above conditions, mentioned in [17], in the following theorem

Theorem 1: Let $X \sim$ PITL (α_1, θ_1) and $Y \sim$ PITL (α_2, θ_2). If $\alpha_1 > \alpha_2$ and $\theta_1 > \theta_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$, and $X \leq_{sr} Y$.

Proof

It is sufficient to show $f_X(x)/f_Y(x)$ is a decreasing function of x; the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\theta_1 x^{2\theta_1-1}\left(1+x^{\theta_1}\right)^{2\alpha_1-1}\left(1+2x^{\theta_1}\right)^{\alpha_1-1}}{\alpha_2\theta_2 x^{2\theta_2-1}\left(1+x^{\theta_2}\right)^{2\alpha_2-1}\left(1+2x^{\theta_2}\right)^{\alpha_2-1}}. \tag{24}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} &= \frac{2\theta_1-2\theta_2}{x} + \frac{(2\alpha_1-1)\,\theta_1 x^{\theta_1-1}}{1+x^{\theta_1}} - \frac{(2\alpha_2-1)\,\theta_2 x^{\theta_2-1}}{1+x^{\theta_2}} \\ &\quad + \frac{2(\alpha_1-1)\,\theta_1 x^{\theta_1-1}}{1+2x^{\theta_1}} - \frac{2(\alpha_2-1)\,\theta_2 x^{\theta_2-1}}{1+2x^{\theta_2}} < 0. \end{aligned} \tag{25}$$

Thus, $f_X(x)/f_Y(x)$ is decreasing in x and hence $X \leq_{lr} Y$. Similarly, we can conclude that for $X \leq_{hr} Y, X \leq_{mrl} Y, X \leq_{sr} Y$.

4 Acceptance Sampling Plans

We assume that the lifetime of a product follows the PITL distribution with parameters (α, θ) defined by (4) and the specified median lifetime of the units claimed by a producer is m_0. Our interest is to make an inference about the acceptance or rejection of the proposed lot based on the criterion that the actual median lifetime, m, of the units is larger than the prescribed lifetime m_0. A common practice in life testing is to terminate the life test by a pre-determined time t_0 and note the number of failures. Now to observe median lifetime, the experiment is run for a $t_0 = am_0$ units of time, multiple of claimed median lifetime with any positive constant a. The idea to accept the proposed lot based on the evidence that $m \geq m_0$, given probability of at least p^*(consumer's risk) using single acceptance sampling plan is as follows [18].

Draw a random sample of n number of units from the proposed lot and conduct an experiment for t_0 units of time. If during the experiment c or less number of units (acceptance number) fail then accept the whole lot, other than the lot is rejected. Observe that probability of accepting a lot, consider sufficiently large sized lots so that the binomial distribution can be applied, under the proposed sampling plan is given by

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 1, \ldots, n \tag{26}$$

Table 1: Single sampling plan for PITL distribution at $\alpha = 0.5, \theta = 1$

p^*	c	$a = 0.25$		$a = 0.5$		$a = 0.75$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.25	0	2	0.7863	1	1	1	1	1	1
	2	9	0.7662	6	0.7578	5	0.7707	4	0.8750
	4	17	0.7557	10	0.8207	9	0.7569	8	0.7734
	6	25	0.7608	15	0.8066	13	0.7603	11	0.8281
	8	33	0.7696	20	0.8031	17	0.7678	15	0.7880
	10	42	0.7529	26	0.7565	21	0.7767	19	0.7597
0.5	0	3	0.6183	2	0.6455	2	0.5592	2	0.5000
	2	13	0.5105	8	0.5221	6	0.6100	6	0.5002
	4	22	0.5238	13	0.5703	11	0.5284	10	0.5001
	6	31	0.5334	19	0.5331	15	0.5740	14	0.5001
	8	41	0.5080	25	0.5073	20	0.5267	18	0.5002
	10	50	0.5179	30	0.5416	24	0.5635	22	0.5001
0.75	0	6	0.3007	4	0.2690	3	0.3127	3	0.2500
	2	18	0.2632	11	0.2521	8	0.3346	7	0.3437
	4	29	0.2555	17	0.2766	13	0.3276	12	0.2744
	6	39	0.2688	23	0.2871	19	0.2503	16	0.3036
	8	50	0.2523	29	0.2918	23	0.3064	21	0.2517
	10	60	0.2576	36	0.2537	28	0.2959	25	0.2706
0.95	0	13	0.0559	7	0.0723	6	0.0547	5	0.0625
	2	28	0.0520	16	0.0576	12	0.0733	11	0.0546
	4	41	0.0511	24	0.0504	18	0.0691	16	0.0592
	6	53	0.0525	31	0.0530	24	0.0608	21	0.0576
	8	65	0.0512	38	0.0525	30	0.0524	26	0.0538
	10	76	0.0544	45	0.0507	35	0.0587	30	0.0680
0.99	0	20	0.0104	11	0.0125	8	0.0171	7	0.0156
	2	37	0.0101	21	0.0109	16	0.0127	14	0.0112
	4	51	0.0105	29	0.0120	23	0.0104	19	0.0154
	6	64	0.0111	37	0.0113	29	0.0111	25	0.0113
	8	77	0.0107	44	0.0128	35	0.0108	30	0.0120
	10	89	0.0113	52	0.0107	41	0.0101	35	0.0121

where $p = F_{PITL}(t_0; \alpha, \theta)$ defined by (3). The function $L(p)$ is the operating characteristic function of the sampling plan, i.e., the acceptance probability of the lot as function of the failure probability. Further using $t_0 = am_0$, thus p_0 can be written as

$$p_0 = 1 - \left\{ \frac{\left(1 + 2\,(ma_0)^{\theta}\right)^{\alpha}}{\left(1 + (ma_0)^{\theta}\right)^{2\alpha}} \right\}. \tag{27}$$

Table 2: Single sampling plan for PITL distribution at $\alpha = 1.5$, $\theta = 1$

p^*	c	$a = 0.25$		$a = 0.5$		$a = 0.75$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.25	0	3	0.7832	1	1	1	1	1	1
	2	16	0.7555	7	0.7935	5	0.8195	4	0.8750
	4	30	0.7641	13	0.7927	9	0.8245	8	0.7734
	6	45	0.7623	20	0.7587	14	0.7685	11	0.8281
	8	61	0.7510	26	0.7802	18	0.7983	15	0.7880
	10	76	0.7592	33	0.7649	23	0.7686	19	0.7597
0.5	0	6	0.5428	3	0.5293	2	0.5988	2	0.5000
	2	23	0.5279	10	0.5380	7	0.5419	6	0.5002
	4	41	0.5063	17	0.5487	12	0.5296	10	0.5001
	6	58	0.5125	25	0.5072	17	0.5234	14	0.5001
	8	76	0.5007	32	0.5212	22	0.5195	18	0.5002
	10	93	0.5065	39	0.5325	27	0.5166	22	0.5001
0.75	0	12	0.2608	5	0.2802	3	0.3586	3	0.2500
	2	34	0.2520	14	0.2690	9	0.3130	7	0.3437
	4	54	0.2559	22	0.2838	15	0.2763	12	0.2744
	6	74	0.2514	31	0.2521	20	0.3045	16	0.3036
	8	93	0.2557	39	0.2550	26	0.2697	21	0.2517
	10	112	0.2570	47	0.2552	31	0.2871	25	0.2706
0.95	0	25	0.0532	10	0.0571	6	0.0770	5	0.0625
	2	53	0.0525	21	0.0606	14	0.0569	11	0.0546
	4	77	0.0537	31	0.0592	20	0.0683	16	0.0592
	6	100	0.0534	41	0.0533	27	0.0546	21	0.0576
	8	123	0.0506	50	0.0547	33	0.0560	26	0.0538
	10	144	0.0527	59	0.0543	39	0.0556	30	0.0680
0.99	0	38	0.0109	15	0.0116	9	0.0165	7	0.0156
	2	70	0.0108	28	0.0112	18	0.0120	14	0.0112
	4	98	0.0101	39	0.0113	25	0.0130	19	0.0154
	6	123	0.0102	49	0.0120	32	0.0121	25	0.0113
	8	147	0.0104	59	0.0117	39	0.0106	30	0.0120
	10	170	0.0107	69	0.0109	45	0.0119	35	0.0121

Now, the problem is to determine for given values of $p^* \, (0 < p^* < 1)\,, am_0$ and c the smallest positive integer n such that

$$L\left(p_0\right) = \sum_{i=0}^{c} \binom{n}{i} p_0^i \left(1 - p_0\right)^{n-i} \leq 1 - p^* \tag{28}$$

where p_0 is given by (27).

Table 3: Single sampling plan for PITL distribution at $\alpha = 0.5$, $\theta = 2$

p^*	c	$a = 0.25$		$a = 0.5$		$a = 0.75$		$a = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
0.25	0	7	0.7715	2	0.7863	1	1	1	1
	2	41	0.7612	9	0.7662	5	0.8434	4	0.8750
	4	80	0.7577	17	0.7557	10	0.7745	8	0.7734
	6	121	0.7540	25	0.7608	14	0.8154	11	0.8281
	8	163	0.7514	33	0.7696	19	0.7925	15	0.7880
	10	205	0.7524	42	0.7529	24	0.7782	19	0.7597
0.5	0	17	0.5008	3	0.6183	2	0.6203	2	0.5000
	2	63	0.5094	13	0.5105	7	0.5865	6	0.5002
	4	111	0.5008	22	0.5238	12	0.5888	10	0.5001
	6	158	0.5024	31	0.5334	18	0.5174	14	0.5001
	8	205	0.5037	41	0.5080	23	0.5330	18	0.5002
	10	252	0.5048	50	0.5179	28	0.5457	22	0.5001
0.75	0	33	0.2508	6	0.3007	3	0.3848	3	0.2500
	2	92	0.2548	18	0.2632	10	0.2720	7	0.3437
	4	148	0.2510	29	0.2555	16	0.2678	12	0.2744
	6	202	0.2502	39	0.2688	22	0.2579	16	0.3036
	8	254	0.2540	50	0.2523	27	0.2943	21	0.2517
	10	307	0.2511	60	0.2576	33	0.2777	25	0.2706
0.95	0	70	0.0506	13	0.0559	7	0.0570	5	0.0625
	2	147	0.0511	28	0.0520	15	0.0545	11	0.0546
	4	214	0.0512	41	0.0511	22	0.0547	16	0.0592
	6	278	0.0501	53	0.0525	29	0.0501	21	0.0576
	8	339	0.0500	65	0.0512	35	0.0564	26	0.0538
	10	398	0.0505	76	0.0544	41	0.0605	30	0.0680
0.99	0	107	0.0102	20	0.0104	10	0.0136	7	0.0156
	2	196	0.0101	37	0.0100	19	0.0128	14	0.0112
	4	271	0.0102	51	0.0105	27	0.0115	19	0.0154
	6	341	0.0100	64	0.0111	34	0.0123	25	0.0113
	8	407	0.0102	77	0.0107	41	0.0121	30	0.0120
	10	472	0.0100	89	0.0113	48	0.0114	35	0.0121

By solving the inequality in (28) for n with given consumer's risk p^*, positive constant a, acceptance number c and p_0, which computed according to parameters (α, θ) and t_0. The solution of the inequality in (28) depends on searching the minimum value of n which makes the left-hand side of the given inequality is less than or equal $1-p^*$.

The minimum values of n satisfying the inequality (28) and its corresponding operating characteristic probability are obtained and displayed in Tabs. 1–3 for the following assumed parameters:

1. $p^* = 0.25, 0.5, 0.75, 0.95, 0.99,\ c = 0(2)8$.
2. $a = 0.25, 0.5, 0.75, 1$ (Note that when $a = 1,\ t_0 = m_0 = 0.5\,\forall\,\alpha, \theta$).
3. $(\alpha, \theta) = (0.5, 1), (1.5, 1), (0.5, 2)$.

From the results obtained in Tabs. 1–3, we notice that:

- With increasing p^*, the required sample size n is increasing.
- With increasing c, the required sample size n is increasing.
- With increasing a, the required sample size n is decreasing.
- With increasing α and fixed θ, the required sample size n is increasing.
- With increasing θ, and fixed α, the required sample size n is increasing.

Finally, for all results checked that $L(p_0) \le 1 - p^*$. Also, when $a = 1$, we have $p_0 = 0.5$, as $t_0 = m_0$ and hence all results $(n,\ L(p_0))$ for any vector of parameter (α, θ) are the same.

5 Parameter Estimation

In this section, the parameter estimation of the PITL distribution is discussed using classical and Bayesian estimation methods. The classical methods include ML, MPS, LS, and WLS.

5.1 ML Estimators

Let $X_1, X_2, \ldots, X_n$ be the observed random sample from the PITL distribution with PDF (4). The log-likelihood function of the PITL distribution, denoted by $\ln \ell$, for parameters, based on complete sample, is given by

$$\ln \ell = n \ln 2\alpha + n \ln \theta - (2\alpha + 1) \sum_{i=1}^{n} \ln\left(1 + x_i^{\theta}\right) + (\alpha - 1) \sum_{i=1}^{n} \ln\left(1 + 2{x_i}^{\theta}\right). \tag{29}$$

The partial derivatives of $\ln \ell$ with respect to α and θ are given by

$$\frac{\partial \ln \ell}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \ln\left(1 + x_i^{\theta}\right) + \sum_{i=1}^{n} \ln\left(1 + 2{x_i}^{\beta}\right), \tag{30}$$

and,

$$\frac{\partial \ln \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{(2\alpha + 1)\, x_i^{\theta} \ln x_i}{\left(1 + x_i^{\theta}\right)} + \sum_{i=1}^{n} \frac{2\,(\alpha - 1)\, x_i^{\theta} \ln x_i}{\left(1 + 2x_i^{\theta}\right)}. \tag{31}$$

The non-linear equations $\partial \ln \ell / \partial \alpha = 0$ and $\partial \ln \ell / \partial \theta = 0$ are solved numerically via iterative technique, to get the ML estimators of α and θ.

5.2 MPS Estimators

A strong alternative procedure, known as MPS, for estimating the population parameters of continuous distributions was proposed in [19]. Let

$$D_i(\alpha,\theta) = F\left(x_{(i)}\,|\alpha,\theta\right) - F\left(x_{(i-1)}\,|\alpha,\theta\right), \quad i = 1,2,\ldots,n+1, \tag{32}$$

be the uniform spacings of a random sample from the PITL distribution, where

$$F\left(x_{(0)}\,|\alpha,\theta\right) = 0, \quad F\left(x_{(n+1)}\,|\alpha,\theta\right) = 1 \text{ and } \sum_{i=1}^{n+1} D_i(\alpha,\theta) = 1. \tag{33}$$

The MPS estimator is obtained by maximizing the geometric mean (**GM**) of the spacings

$$GM(\alpha,\theta) = \left\{\prod_{i=1}^{n+1} D_i(\alpha,\beta,\theta)\right\}^{\frac{1}{n+1}}, \tag{34}$$

with respect to α and θ, or we maximize the logarithm of the GM of sample spacings (34) with respect to α and θ. The numerical technique is used to otain the desired estimators.

5.3 Least Squares and Weighted Least Squares Estimators

Let $X_1, X_2, \ldots, X_n$ is a random sample of size n drawn from the PITL distribution and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the observed ordered sample. The LS estimators are derived by minimizing the sum of squares errors,

$$\sum_{i=1}^{n}\left[F\left(X_{(i)}\right) - \frac{i}{n+1}\right]^2, \tag{35}$$

related to the population parameters. So, the LS estimators of the model parameters of the PITL distribution are obtained by minimizing the following formula

$$\sum_{i=1}^{n}\left[1 - \left\{\frac{\left(1+2x_{(i)}^{\theta}\right)^{\alpha}}{\left(1+x_{(i)}^{\theta}\right)^{2\alpha}}\right\} - \frac{i}{n+1}\right]^2, \tag{36}$$

related to α and θ. Furthermore, the WLS estimators of the PITL distribution is obtained by minimizing the following related to α and θ.

$$\sum_{i=1}^{n}\frac{1}{var\left(F\left(X_{(i)}\right)\right)}\left[1 - \left\{\frac{\left(1+2x_{(i)}^{\theta}\right)^{\alpha}}{\left(1+x_{(i)}^{\theta}\right)^{2\alpha}}\right\} - \frac{i}{n+1}\right]^2, \tag{37}$$

where, $var\left(F\left(X_{(i)}\right)\right) = \dfrac{(n+1)^2(n+2)}{i(n-i+1)}$.

5.4 *Bayesian Estimators*

The Bayesian estimator using squared error loss function (SELF) under the assumption of non-informative prior of the population parameters for PITL distribution is obtained. Assuming the prior distributions of α and θ have uniform density function, where the joint prior PDF are given by

$$\pi\left(\alpha,\theta\right)\propto\left(\alpha\theta\right)^{-1};\quad \alpha,\ \theta>0. \tag{38}$$

The posterior density of α and θ given the data is

$$\pi\left(\alpha,\theta\left|x\right.\right)\propto 2^{n}\left(\alpha\theta\right)^{n-1}\prod_{i=1}^{n}\left[x_i^{(2\theta-1)}\left(1+x_i^{\theta}\right)^{-(2\alpha+1)}\left(1+2x_i^{\theta}\right)^{\alpha-1}\right]. \tag{39}$$

Table 4: The MSE, RB and MCMC for different estimates of the PITL distribution

Parameter	MLE		LSE		WLSE		MPSE		BE		
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MCMC
$n=20$											
$\alpha=0.5$	0.0502	3.3614*	0.0874	0.1339	0.0756	0.1291	0.0774	0.4803	0.0221	0.2970	0.1285*
$\theta=0.25$	0.0557	0.3658	0.0423	0.1941	0.0238	0.1440	0.1694	1.3526	0.0220	0.5932	0.5047*
$\alpha=0.5$	0.0477	0.0886	0.0820	0.0801	0.0665	0.0685	0.1669	0.1159	0.0394	0.3968	0.1487*
$\theta=0.5$	0.7593	0.5961	1.3296	0.4823	0.1710	0.1884	0.0523	0.0832	0.0514	0.4526	1.2930*
$n=50$											
$\alpha=0.5$	0.0159	0.0255	0.0327	0.0265	0.0246	0.0113	0.0714	0.5178	9.9297*	0.1992	0.1221*
$\theta=0.25$	7.3359*	0.0927	0.0106	0.0761	8.7961*	0.0701	0.0946	1.1191	6.3198*	0.3176	0.3541*
$\alpha=0.5$	0.0199	1.6015*	0.0351	0.0832	0.0285	0.0627	0.0202	0.0268	0.0143	0.2392	0.1297*
$\theta=0.5$	0.0371	0.1106	0.0752	0.0591	0.0570	0.0632	0.0220	0.0882	0.0193	0.2768	0.9988*
$n=75$											
$\alpha=0.5$	0.0142	0.0407	0.0257	0.0336	0.0196	0.0156	0.0639	0.4961	6.3469*	0.1592	0.1281*
$\theta=0.25$	4.1856*	0.1063	7.0371*	0.0699	4.9633*	0.0675	0.0818	1.0437	3.5326*	0.2364	0.5969*
$\alpha=0.5$	0.0125	0.0366	0.0220	0.0481	0.0159	0.0191	0.0106	8.4056*	9.9320*	0.1992	0.1221*
$\theta=0.5$	0.0166	0.0860	0.0169	0.0139	0.0139	0.0309	0.0124	0.0246	0.0108	0.2070	0.9626*
$n=100$											
$\alpha=0.5$	0.0112	0.0104	0.0162	0.0282	0.0125	0.0193	0.0606	0.5129	2.4800*	0.0994	0.1367*
$\theta=0.25$	2.3740*	0.0435	3.9248*	0.0181	2.7331*	0.0166	0.0784	1.0736	1.5718*	0.1572	0.4978*
$\alpha=0.5$	9.9583*	0.0163	0.0148	0.0370	0.0117	0.0159	5.8392*	0.0207	4.8690*	0.1394	0.1220*
$\theta=0.5$	7.8253*	0.0520	9.0235*	5.8805*	7.5677*	0.0202	0.0105	0.0354	6.2709*	0.1572	0.8880*
$n=150$											
$\alpha=0.5$	6.3963*	8.7789*	0.0116	0.0137	8.3495*	0.8879*	0.0565	0.5124	0.3927*	0.0392	0.1211*
$\theta=0.25$	1.0688*	0.0262	2.5472*	0.0109	1.4929*	0.0166	0.0762	1.0570	0.3908*	0.0760	0.5356*
$\alpha=0.5$	4.9217*	0.0156	0.0123	0.0282	8.4621*	0.0206	4.2623*	0.0486	2.4703*	0.0992	0.1189*
$\theta=0.5$	4.3257*	0.9536*	8.3465*	0.0884*	6.8160*	1.4037*	5.5084*	0.0513	3.5151*	0.1170	0.9037*

Note: *Indicate that the value multiply 10^{-3}.

Therefore, the Bayesian estimators of $\upsilon = (\alpha, \theta)$ under SELF; denoted by $\tilde{q}_{(SELF)}(\upsilon)$ can be calculated as follows:

$$\tilde{q}_{(SELF)}(\upsilon) = \int_0^{\infty}\int_0^{\infty} \upsilon L(\alpha, \theta \,|x)\, \pi(\alpha, \theta \,|x)\, d\alpha d\theta. \tag{40}$$

Generally, the ratio of two integrals given by (40) cannot be obtained in a closed form. Then, the integral Eq. (40) is solved numerically due to its complicated forms.

6 Monte Carlo Simulation

As mentioned in previous section, expressions for the derived estimators are hard to obtain. Therefore, we design simulation study for clarifying the theoretical results. The behavior of estimates is examined in terms of their mean square error (**MSE**), and relative bias (**RB**). We perform the following:

Step 1: 10000 random samples of sizes 20, 50, 75 and 150 are generated from PITL distribution. The chosen parameters values are;

$(\alpha = 0.5, \theta = 0.25)$, $(\alpha = 0.5, \theta = 0.5)$, $(\alpha = 0.5, \theta = 0.75)$, $(\alpha = 0.5, \theta = 1.25)$, $(\alpha = 0.25, \theta = 0.25)$, $(\alpha = 0.75, \theta = 0.25)$, $(\alpha = 1.25, \theta = 0.25)$, $(\alpha = 1.5, \theta = 0.25)$.

Table 5: The MSE, RB and MCMC for different estimates of the PITL distribution

Parameter	MLE		LSE		WLSE		MPSE		BE		
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MCMC
$n = 20$											
$\alpha = 0.5$	0.0499	0.0332	0.0884	0.1528	0.0766	0.1549	0.2042	0.4776	0.0393	0.3962	0.1950*
$\theta = 0.75$	0.4311	0.2953	0.3510	0.1592	0.1739	0.0948	0.0756	0.1948	0.0710	0.3547	1.4330*
$\alpha = 0.5$	0.0599	4.9900*	0.1039	0.2159	0.0896	0.1743	1.2878	1.8693	0.0521	0.4562	0.1645*
$\theta = 1.5$	1.0838	0.2532	0.4214	4.3805*	0.4393	0.0255	0.8898	0.6065	0.4078	0.4247	4.3470*
$n = 50$											
$\alpha = 0.5$	0.0176	0.0315	0.0351	0.0305	0.0268	9.7007*	0.0968	0.4744	0.0143	0.2394	0.1143*
$\theta = 0.75$	0.0703	0.1141	0.0946	0.0823	0.0797	0.0838	0.0680	0.2982	0.0516	0.3019	1.7770*
$\alpha = 0.5$	0.0170	0.0263	0.0377	0.1040	0.0281	0.0557	0.9997	1.8786	0.0143	0.2392	0.1344*
$\theta = 1.5$	0.1821	0.0682	0.2046	0.0331	0.1788	0.2410*	0.8582	0.6468	0.1559	0.2620	3.6850*
$n = 75$											
$\alpha = 0.5$	0.0126	0.0101	0.0232	0.0443	0.0174	0.0334	0.0729	0.4526	9.9517*	0.1994	0.1222*
$\theta = 0.75$	0.0310	0.0574	0.0615	0.0357	0.0356	0.0246	0.0633	0.3001	0.0219	0.1961	1.6290*
$\alpha = 0.5$	9.5928*	0.0223	0.0179	0.0408	0.0137	0.0155	0.9882	1.9169	8.9794*	0.1894	0.1218*
$\theta = 1.5$	0.1405	0.0568	0.1502	0.0823*	0.1322	0.0185	0.6718	0.6539	0.1063	0.2160	3.4330*
$n = 100$											
$\alpha = 0.5$	0.0125	5.1640*	0.0174	0.0508	0.0143	0.0300	0.0657	0.4518	7.1852*	0.1694	0.1127*
$\theta = 0.75$	0.0195	0.0390	0.0292	3.4620*	0.0241	0.0119	0.0617	0.2958	0.0141	0.1568	1.4390*
$\alpha = 0.5$	8.9568*	0.0326	0.0177	0.0562	0.0124	6.3811*	0.8697	1.8196	8.5678*	0.1850	0.1359*
$\theta = 1.5$	0.0508	4.6253*	0.1026	5.8072*	0.0976	7.6945*	0.6288	0.6394	0.0352	0.1233	3.0300*
$n = 150$											
$\alpha = 0.5$	5.6017*	0.0428	0.0102	0.0188	7.3563*	0.0319	0.0612	0.4535	4.8689*	0.1394	0.1177*
$\theta = 0.75$	0.0108	0.0513	0.0183	0.0320	0.0127	0.0390	0.0571	0.3003	9.7996*	0.1305	1.3600*
$\alpha = 0.5$	7.4829*	0.0259	0.0122	0.0366	0.0104	0.0146	0.7230	1.8869	5.5761*	0.1492	0.1344*
$\theta = 1.5$	0.0380	9.9734*	0.0682	4.6852*	0.0744	0.0260	0.4469	0.6462	0.0249	0.1033	2.7410*

Table 6: The MSE, RB and MCMC for different estimates of the PITL distribution

Parameter	MLE		LSE		WLSE		MPSE		BE		
	MSE	**RB**	**MSE**	**RB**	**MSE**	**RB**	**MSE**	**RB**	**MSE**	**RB**	**MCMC**
$n = 20$											
$\alpha = 0.25$	0.0215	0.0175	0.0556	0.4366	0.0371	0.3402	0.3629	0.4150	0.0193	0.5556	0.0881*
$\theta = 0.25$	0.1610	0.6416	0.0329	0.0167	0.0247	0.0102	0.0331	0.2617	0.0221	0.5940	0.3558*
$\alpha = 0.75$	0.0834	0.0294	0.1045	0.0356	0.0936	0.0213	0.2654	0.6785	0.0768	0.3693	0.2208*
$\theta = 0.25$	0.0128	0.1919	0.0184	0.1138	0.0202	0.1319	0.4077	2.2835	0.0118	0.4340	0.4083*
$n = 50$											
$\alpha = 0.25$	9.4403*	0.0566	0.0180	0.2453	0.0117	0.1394	9.6211*	5.4851*	8.9721*	0.3788	0.0761*
$\theta = 0.25$	0.0279	0.2695	0.0150	0.0508	0.0117	4.4851*	0.0105	0.1048	0.0101	0.4020	0.2627*
$\alpha = 0.75$	0.0423	0.2867*	0.0588	0.0319	0.0502	0.0231	0.2610	0.6774	0.0398	0.2657	0.1910*
$\theta = 0.25$	4.3629*	0.0711	4.9722*	0.0340	4.1783*	0.0372	0.3376	2.1739	3.8803*	0.2484	0.4547*
$n = 75$											
$\alpha = 0.25$	6.2262*	2.9251*	0.0162	0.1852	9.7917*	0.0924	5.6048*	0.0443	5.3033*	0.2912	0.0692*
$\theta = 0.25$	0.0101	0.1020	9.9630*	0.0252	6.4655*	0.0121	7.5809*	0.1188	6.3290*	0.3180	0.2473*
$\alpha = 0.75$	0.0203	5.2508*	0.0306	0.0491	0.0242	0.0285	0.2590	0.6766	0.0167	0.1723	0.1830*
$\theta = 0.25$	1.9143*	0.0407	2.9400*	7.9264*	2.1453*	6.5916*	0.2964	2.1037	1.5651*	0.1572	0.4263*
$n = 100$											
$\alpha = 0.25$	3.3672*	0.0541	9.3865*	0.0411	5.0419*	8.1889*	2.9053*	0.0122	2.4833*	0.1992	0.0658*
$\theta = 0.25$	4.7421*	0.1040	9.7320*	0.0128	5.6535*	0.0415	3.4359*	0.0356	3.1820*	0.2244	0.5566*
$\alpha = 0.75$	0.0139	0.0273	0.0194	3.4386*	0.0198	0.0179	0.2517	0.6675	0.0103	0.1355	0.1723*
$\theta = 0.25$	1.4259*	0.0554	2.5388*	0.0329	1.8250*	0.1215	0.2674	2.0158	1.1283*	0.1332	0.4106*
$n = 150$											
$\alpha = 0.25$	2.5287*	0.0270	5.7767*	0.0914	3.5442*	0.0558	1.9448*	0.0507	1.8268*	0.1708	0.0658*
$\theta = 0.25$	2.4121*	0.0266	4.9991*	0.8099*	2.4142*	4.2659*	3.0967*	0.0946	2.2711*	0.1896	0.5666*
$\alpha = 0.75$	0.0111	0.0161	0.0179	0.0141	0.0183	0.0350	0.2495	0.6487	9.5698*	0.1303	0.1609*
$\theta = 0.25$	1.0226*	0.0312	1.6564*	0.0373	1.3509*	0.0343	0.2621	1.9717	1.0077*	0.1260	0.3476*

Note: *Indicate that the value multiply 10^{-3}.

Step 2: ML estimate (**MLE**), MPS estimate (**MPSE**), LS estimate (**LSE**), WLS estimate (**WLSE**) and Bayes estimate (**BE**) of the parameters are obtained.

Step 3: Markov Chain Monte Carlo (**MCMC**) technique (as M-H algorithm) is used to get the BEs of α and θ under SELF via 10000 iterations.

Step 4: Compute MSEs and RBs of all estimates and the results are listed in Tabs. 4–7. We notice the following about the performance of estimates:

- For all methods of estimation, it is clear that MSEs and RBs decrease as n gets larger for all parameters (see Tabs. 4–7).
- The MSEs of BS are the less than the corresponding for other methods in almost all cases (see Tabs. 4–7).
- For fixed value of α and as the value of θ gets larger, the MSEs and RBs of θ estimates are increasing for different methods (see Tabs. 4–7).
- For fixed value of α and as the value of θ decreases, the MSEs and RBs of θ estimates are decreasing for different methods (see Tabs. 4–7).

- For fixed value of θ and as the value of α increases, the MSEs and RBs of α estimates increase based on the different methods. But the MSEs and RBs of estimates of θ decrease for different methods (see Tabs. 4–7).
- For fixed value of θ and as the value of α decreases, the MSEs and RBs of α estimates decrease based on the different methods. While the MSEs and RBs of θ estimates for different methods increase (see Tabs. 4–7).

Table 7: The MSE, RB and MCMC for different estimates of the PITL distribution

Parameter	MLE		LSE		WLSE		MPSE		BE		
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MCMC
$n = 20$											
$\alpha = 1.25$	0.1316	0.0610	0.1259	0.0101	0.1181	0.0178	1.0292	0.8094	0.1064	0.2608	0.3346*
$\theta = 0.25$	0.0118	0.1403	7.6849*	0.0231	6.4685*	0.0313	1.1295	4.0226	6.1965*	0.3144	0.3638*
$\alpha = 1.5$	0.1962	4.4106*	0.2397	0.0282	0.2205	0.0270	1.5919	0.8382	0.1562	0.2633	0.4202*
$\theta = 0.25$	3.4040*	0.0418	5.7670*	5.4885*	4.6340*	0.0111	1.5220	4.6453	3.2419*	0.2272	0.3422*
$n = 50$											
$\alpha = 1.25$	0.0713	6.3716*	0.0938	7.9533*	0.0837	3.4523*	1.0243	0.8091	0.0698	0.2112	0.2912*
$\theta = 0.25$	2.1955*	0.0225	2.4607*	0.0157	2.0882*	5.3997*	1.1227	4.1105	2.0692*	0.1812	0.3720*
$\alpha = 1.5$	0.0698	3.1267*	0.0887	4.2921*	0.0809	3.6117*	1.5569	0.8314	0.0621	0.1660	0.3352*
$\theta = 0.25$	1.6055*	0.0374	2.0401*	8.7988*	1.7800*	0.0151	1.3746	4.5562	1.5608*	0.1572	0.3621*
$n = 75$											
$\alpha = 1.25$	0.0448	0.0136	0.0667	0.0281	0.0584	0.0228	1.0129	0.8068	0.0438	0.1672	0.2738*
$\theta = 0.25$	1.0791*	0.0227	1.6382*	6.5046*	1.3702*	0.0121	1.0994	4.1070	1.0719*	0.1300	0.3523*
$\alpha = 1.5$	0.0449	5.0528*	0.0615	1.4778*	0.0585	0.9863*	1.5565	0.8314	0.0438	0.1393	0.3279*
$\theta = 0.25$	0.9677*	0.0107	1.7814*	1.5020*	1.5890*	8.4333*	1.3203	4.4979	0.9455*	0.1220	0.3482*
$n = 100$											
$\alpha = 1.25$	0.0316	9.9876*	0.0399	0.0173	0.0352	8.0360*	1.0107	0.8040	0.0293	0.1368	0.2822*
$\theta = 0.25$	0.8071*	0.0250	0.9559*	8.3144*	0.8334*	3.0754*	1.0455	4.0291	0.7656*	0.1096	0.3399*
$\alpha = 1.5$	0.0435	0.0211	0.0448	0.0236	0.0429	0.0242	1.5417	0.8275	0.0401	0.1333	0.2025*
$\theta = 0.25$	0.6372*	9.5575*	0.7242*	6.2625*	0.6186*	4.4455*	1.2907	4.4391	0.6115*	0.0980	0.2882*
$n = 150$											
$\alpha = 1.25$	0.0235	0.0135	0.0309	0.0127	0.0281	0.0136	1.0093	0.8035	0.0194	0.1112	0.1630*
$\theta = 0.25$	0.5687*	0.0103	0.8431*	7.4235*	0.7212*	7.2200*	1.0418	4.0430	0.5634*	0.0940	0.2869*
$\alpha = 1.5$	0.0316	9.7917*	0.0354	2.7240*	0.0329	5.0414*	1.4824	0.8059	0.0286	0.1127	0.2191*
$\theta = 0.25$	0.4705*	0.0139	0.6606*	0.0111	0.5335*	0.0136	1.2206	4.1920	0.4353*	0.0824	0.2833*

Note: *Indicate that the value multiply 10^{-3}.

7 Real Data Modelling

This section gives applications of the PITL model using two real data sets. The fits of the PITL distribution, for the first data, is compared with Type II Topp–Leone inverse Rayleigh (**TIITLIR**) [20], inverse Weibull (**IW**), exponentiated generalized power function (**EGPF**) [21], power function (**PF**), and the inverse exponential (**IE**) distributions. On the other hand, the fits of the PITL distribution, for the second data, is compared with TIITLIR, IW, Kumaraswamy Weibull Lomax (**KWL**) [22], inverse Rayleigh (**IR**) and Lomax (**L**) distributions. Criteria is handled to inspect the distribution for best fit: Akaike information criteria (**AIC**), consistent AIC (**CAIC**), Bayesian information criteria (**BIC**), Hannan and Quinn information criteria (**HQIC**). Also, we provide the Kolmogorov Smirnov (**KS**) statistic and the P value. The engineering application of selecting and comparing different distributions in composite structures can be found in [23].

Table 8: The MLEs and SEs of the model parameters and goodness of fit measures for first data

Model	MLE	SE	-2logL	AIC	BIC	CAIC	HQIC	K-S	*P*-value
PITL	$\hat{\alpha}=2.1897$	0.4268	111.09	115.09	118.14	115.45	116.13	0.0781	0.922
	$\hat{\theta}=0.9478$	0.1420							
TIITLIR	$\hat{\alpha}=0.1440$	0.0240	127.42	131.42	134.47	131.78	132.46	0.9375	0.062
	$\hat{\theta}=0.2860$	0.0550							
IW	$\hat{\theta}=0.6173$	0.1277	117.25	121.25	124.31	121.64	122.29	0.2255	0.775
	$\hat{\beta}=0.8804$	0.1093							
EGPF	$\hat{\theta}=0.4201$	0.3262	120.75	128.75	134.86	130.13	130.83	0.9454	0.055
	$\hat{\alpha}=0.9021$	0.2470							
	$\hat{\beta}=1.0712$	0.9076							
	$\hat{\lambda}=8$	–							
PF	$\hat{\alpha}=0.5031$	0.0863	120.95	124.95	128.00	125.34	125.99	0.9327	0.067
	$\hat{\lambda}=8$	–							
IE	$\hat{\theta}=0.5725$	0.0982	118.39	120.39	121.91	120.51	120.91	0.5456	0.454

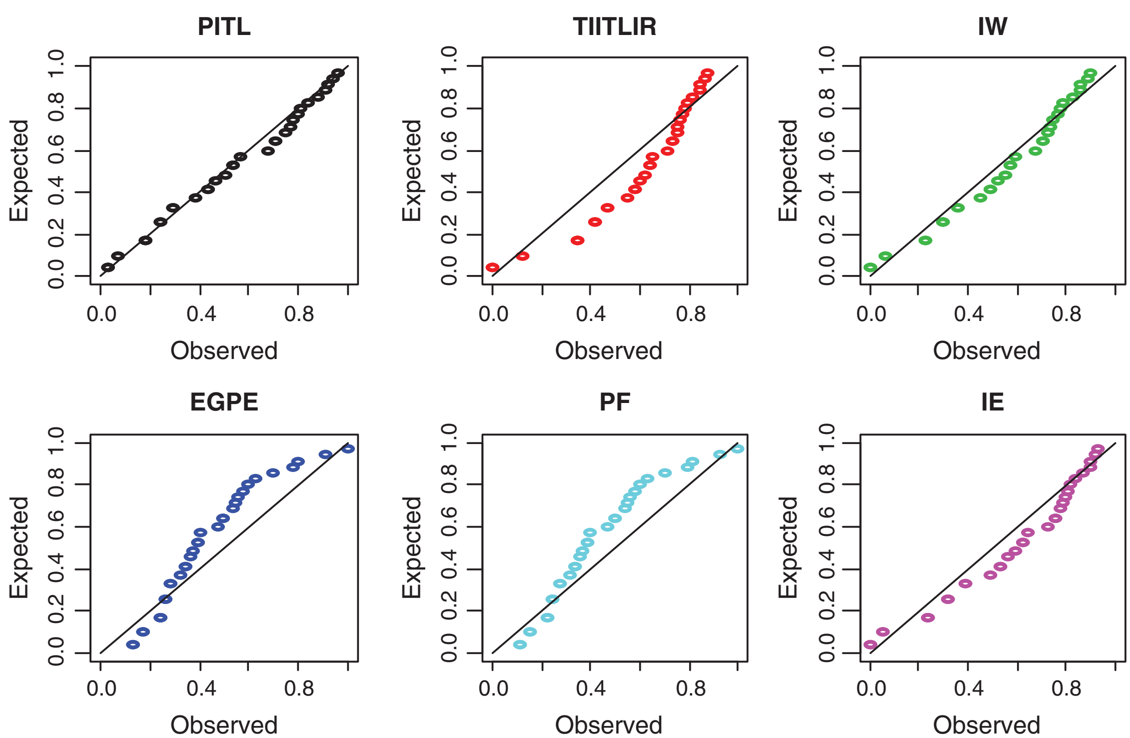

Figure 2: The PP plots of PITL, TIITLIR, IW, EGPF, PF and IE distributions for first data set

First Data: The data set contains 34 observations of the vinyl chloride data obtained from [24] which represents clean up gradient ground–water monitoring wells in mg/L. Tab. 8 gives measures of comparison for the various distributions under study. Also, it contains the MLE and the corresponding standard error (SE) for parameters of each model. Plots of estimated PDF and CDF are given in Fig. 3. The PP plots of estimated densities are given in Fig. 2.

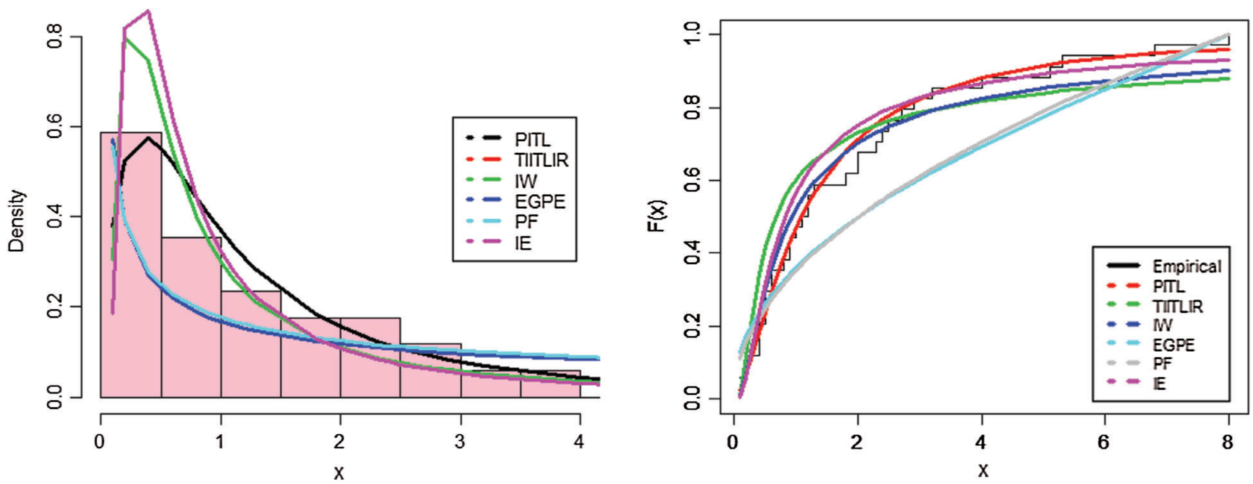

Figure 3: Estimated PDF and CDF of the models for the first data set

Table 9: The MLEs and SEs of the model parameters and goodness of fit measures for second data

Model	MLE	SE	-2logL	AIC	BIC	CAIC	HQIC	K-S	*P*-value
PITL	$\hat{\alpha} = 1.3156$	0.3221	78.66	82.66	85.46	83.07	83.56	0.204	0.796
	$\hat{\theta} = 1.9736$	0.3652							
TIITLIR	$\hat{\alpha} = 0.5860$	0.0790	86.40	90.40	93.21	90.82	91.30	0.812	0.188
	$\hat{\theta} = 0.7310$	0.1720							
IW	$\hat{\theta} = 1.0162$	0.1272	83.83	87.83	90.64	88.28	88.73	0.511	0.489
	$\hat{\beta} = 1.5496$	0.2027							
KWL	$\hat{\alpha} = 0.7828$	0.1472	87.95	95.95	101.55	97.43	97.74	0.816	0.184
	$\hat{\beta} = 0.0354$	0.0722							
	$\hat{\lambda} = 22.7034$	38.7474							
	$\hat{\theta} = 1.9621$	0.5633							
L	$\hat{\lambda} = 100.5541$	227.4926	135.79	139.79	142.60	140.21	140.69	0.930	0.069
	$\hat{\theta} = 167.2669$	379.617							
IR	$\hat{\theta} = 0.9267$	0.0846	88.273	90.273	91.674	90.416	90.721	0.936	0.064

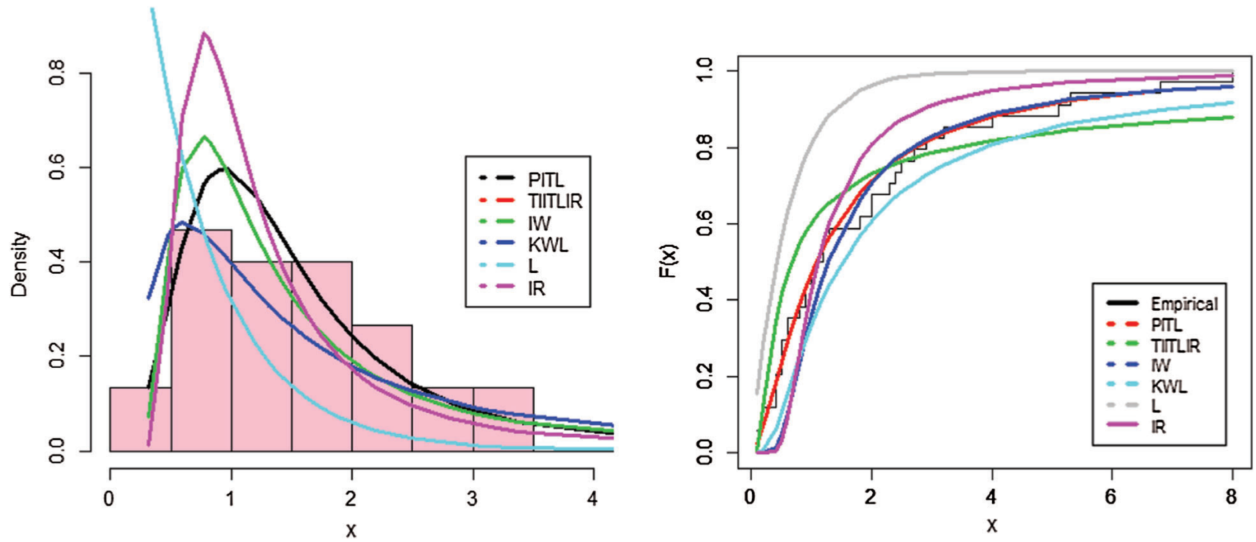

Figure 4: Estimated PDF and CDF of the models for the second data set

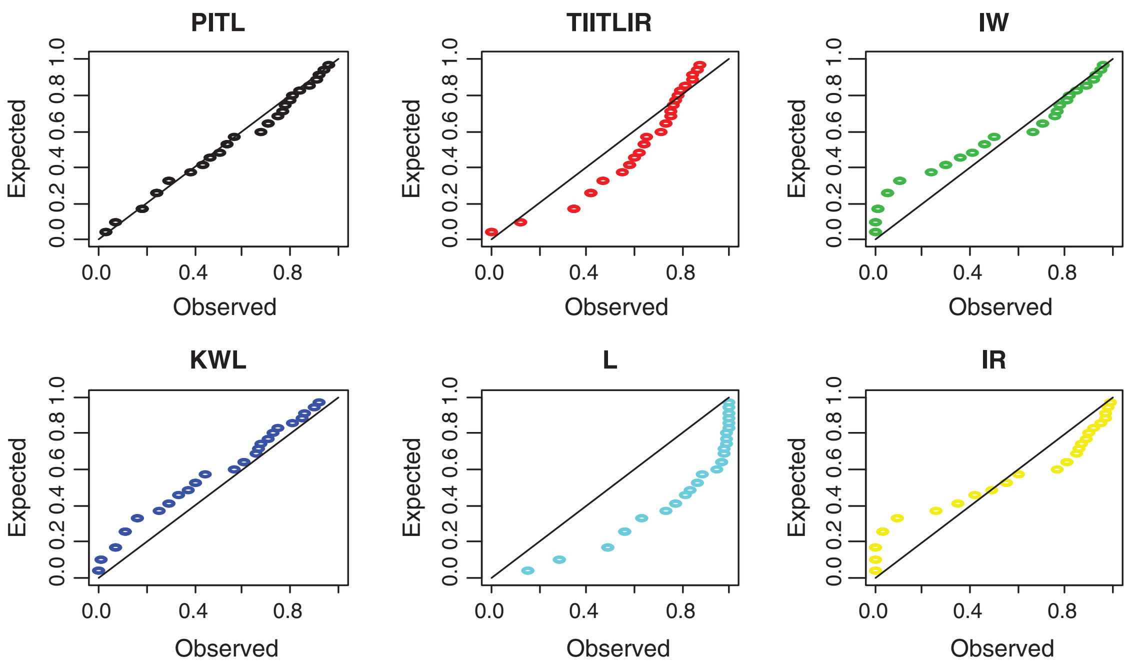

Figure 5: The PP plots of PITL, TIITLIR, IW, KWL, L and IR distributions for second data set

Second Data: The second real-life data was originally reported in [25]. The data contain 30 observations of the March precipitation (in inches) in Minneapolis/St Paul. The observed values are:

0.77, 1.74, 0.81, 1.20, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05.

Tab. 9 provides comparison measures for the proposed distributions. Plots of estimated PDF and CDF of the PITL model are given in Fig. 4. PP plots of estimated densities are given in Fig. 5.

Based on Tabs. 8 and 9 and Figs. 2–5, it can be seen that the PITL provides the overall best fit. Consequently, the PITL distribution can be chosen as suitable model when comparing to other distributions to explain the studied data.

8 Concluding Remarks

In this study, the power inverted Topp–Leone distribution is proposed. It provides more flexibility compared with the inverted Topp–Leone model. Some useful statistical properties of the PITL distribution are provided. We obtain the acceptance sampling plans for the PITL distribution when the life test is truncated at the median life of the stated distribution. At different parameters of the PITL distribution and different levels of consumer's risk, the minimum sample size is computed under multiple truncation times. Also, at the obtained sample sizes, the probability of acceptance is computed to ensure that it's less than or equal the complement of the consumer's risk $(1 - P^{*})$. The model parameters are estimated by the maximum likelihood, maximum product spacing, least squares, weighted least squares and Bayesian methods. A simulation study reveals that the estimates have desirable properties such as small relative biases and mean square errors as sample sizes increase. Then, we deal with two real data application and mention that the PITL model is the better than other competitive distributions.

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