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Approximation of Unit-Hypercubic Infinite Noncooperative Game Via Dimension-Dependent Samplings and Reshaping the Player's Payoffs into Line Array for the Finite Game Simplification

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Abstract: The problem of solving infinite noncooperative games approximately is considered. The game may either have solution or have no solution. The existing solution may be unknown as well. Therefore, an approach of obtaining the approximate solution of the infinite noncooperative game on the unit hypercube is suggested. The unit-hypercubic game isomorphism to compact infinite noncooperative games allows to disseminate the approximation approach on a pretty wide class of noncooperative games. The approximation intention is in converting the infinite game into a finite one, whose solution methods are easier rather than solving infinite games. The conversion starts with sampling the players' payoff functions. Each dimension of the player's pure strategies unit hypercube is sampled with its own sampling constant, being the number of equal-measure intervals between the selected points along the dimension. There are stated requirements for the sufficiently accurate sampling. Having got the finite game on hypercubic lattice after the sampling, every player's payoff multidimensional matrix is reshaped to reduce number of its dimensions down to the number of players. Dimensionality reduction will commonly accelerate computations, connected with the approximate solution consistency. The introduced consistency mechanism rejects the finite game solution, pretended to being the initial game approximate solution, if the solution depends vastly on the sampling steps. If the solution is weakly consistent then, changing the sampling steps minimally, there are non-decreasing difference of the players' payoffs and difference of the players' equilibrium strategies and cardinalities of their supports. If the solution is consistent then the non-decreasing property holds stricter for cardinalities of the supports and their upper densities.

Keywords: Infinite noncooperative game; Unit hypercube; Multidimensional matrix; Strategy support; Approximate solution consistency, Game isomorphism.

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1 Problem of infinite noncooperative games solution

Any infiniteness in game modeling is hard and heavy to handle it. On the one hand, it is difficult to solve infinite noncooperative games analytically. On the other hand, often the infinite game solution appears impossible for its full practical realization, because an infinite support of the equilibrium strategy is practiced only over a zero-measure subset of objects. Moreover, any of these objects (pure strategies) cannot be taken for infinite number of times. Therefore noncooperative game approximation aims at two goals: to solve the game easier and to implement the solution as fast as possible [Osborne (2003)]. And thus easy-and-fast solution is standing against the genuine solution. The matter is that some of important features of the genuine solution may be lost after approximation. It is crucial in rational resources allocation problems [Gąsior and Drwal (2013); Ye and Chen (2013)] with two sides of interest, military processes organization and jurisprudence, involving a few players [Suzdal (1976); Calvert, McCubbins, and Weingast (1989)], socio-economic and ecological gaming models with more players [Friedman (1998); Vorob'yov (1985)], multiagent modeling [Fujii, Yoshimura, and Seki (2010)], multiobjective optimization under uncertainty [Amaziane, Naji, Ouazar, and Cheng (2005); Li and Li (2010); Li, Li, Sun, Luo, and Zhang (2010); Li, Zhao, and Ni (2013); Zhu, Liu, Wang, and Yu (2004); Li, Luo, and Sun (2011); Trapani, Kipouros, and Savill (2012)], etc. So, there is needed a balanced way in searching easy-and-fast solution by approximating infinite noncooperative games.

2 Solving infinite noncooperative games approximately

Not every approximate solution of infinite noncooperative game implies the players' finite supports [Bernhard and Shinar (1990); Chakrabarti (1999)]. Solving approximately can be either a method of thorough manipulation with infinite noncooperative game [Reny (1999); Mallozzi, Pusillo, and Tijs (2008)] or mapping this game into finite one [Zielonka (1998); McNaughton (1993)], whereupon it may be solved as approximately as well as exactly (analytically). Surely, getting the finite game is strongly preferable. However, there is no general theory of solving finite noncooperative games. It even is unclear what this theory could have been. Solving a noncooperative game always is springing up as an original problem, needing usually specific reasonings [Osborne (2003); Vorob'yov (1985); Kostreva and Kinard (1991); Pavel (2007)]. Surely, these reasonings depend on the type of equilibrium, including utility, stability, or equity. Particularly, finding Nash equilibrium solutions in even the finite noncooperative game bears a computational difficulty [Osborne (2003); Vorob'yov (1985); Han, Zhang, Qian, and Xu (2012); Chen and Deng (2007)]. Just when the player has minimal number of pure strategies, and there are no more than three players, the solution is analytical and there is a known technique of its visualization [Vorob'yov (1985); Vorob'yov (1984); Romanuke (2010)]. Thus, solving dyadic games with three players is visualized on the cube of situations in mixed strategies, whereas dyadic games with four players and more are solved purely in analytics, requiring more computational resources [Vorob'yov (1984); Browning and Colman (2004)]. But finite noncooperative games with greater numbers of pure strategies at their players (three and more) are much harder to solve them [Osborne (2003); Han, Zhang, Qian, and Xu (2012); Vorob'yov (1984); Nisan, Roughgarden, Tardos, and Vazirani (2007)]. Nonetheless, difficulties in solving infinite noncooperative games are hardly comparable to those ones while a finite game is solved.

Compact games, having solutions at least in mixed strategies for measurable payoff functions [Osborne (2003); Vorob'yov (1985); Vorob'yov (1984); Kukushkin (2011); Stoltz and Lugosi (2007)], cannot be solved by a universal algorithmic approach, unless they are finite games. But compact games have ε-equilibrium situations with finite supports. Therefore they are reduced via constructing ε-nets [Vorob'yov (1984); Giannopoulos, Knauer, Wahlström, and Werner (2012)] for some $\varepsilon > \varepsilon_0$, where $\varepsilon_0 > 0$ is a minimal distance between two strategies of the player by Helly metric [Vorob'yov (1984)]. However, it is unknown how to select ε_0. An overestimated value ε_0 drives to rough approximation, where some of important features of the genuine solution may be lost. And too small value ε_0 provokes either long-convergent method for approximate solution of the finite game or enormous computational spendings to get that finite game exact solution.

3 Tasks for the goal attainment

Let there be the noncooperative game with $N \in \mathbb{N} \backslash \{1\}$ players. And let n-th player act within the unit M_n-dimensional hypercube

$$H_n = \bigtimes_{m=1}^{M_n} [0;\, 1] \subset \mathbb{R}^{M_n} \text{ by } M_n \in \mathbb{N} \text{ and } n = \overline{1, N}. \tag{1}$$

Thus pure strategy of n-th player is M_n-dimensional point

$$\mathbf{X}_n = [x_{nm}]_{1 \times M_n} \in H_n$$

and in the situation

$$\mathbf{X} = \{\mathbf{X}_n\}_{n=1}^{N} \in \bigtimes_{n=1}^{N} H_n = \bigtimes_{d=1}^{\sum_{i=1}^{N} M_i} [0;\, 1] \subset \mathbb{R}^{\sum_{i=1}^{N} M_i} \tag{2}$$

n-th player gets the payoff $K_n(\mathbf{X})$. Consequently, each of the measurable and bounded payoff functions $\{K_n(\mathbf{X})\}_{n=1}^{N}$ is defined on the unit $(\sum_{i=1}^{N} M_i)$-dimensional hypercube

$$H = \bigtimes_{n=1}^{N} H_n = \bigtimes_{d=1}^{\sum_{i=1}^{N} M_i} [0;\, 1] \subset \mathbb{R}^{\sum_{i=1}^{N} M_i}. \tag{3}$$

And the tuple

$$\left\langle \{H_n\}_{n=1}^{N}, \{K_n(\mathbf{X})\}_{n=1}^{N} \right\rangle \tag{4}$$

is the noncooperative game on hypercube (3). This game is isomorphic [Osborne (2003); Vorob'yov (1985); Vorob'yov (1984)] to noncooperative games, defined on compact subspaces in $\mathbb{R}^{\sum_{n=1}^{N} M_n}$, whereon payoff functions are measurable and bounded. The goal is to convert the infinite noncooperative game (4) on unit hypercube (3) into a finite game, whose formal representation shall be as simple as possible. The finite game solution of a certain type should reflect all significant features of the game (4) genuine solution. In this way the game (4) is going to be approximated, overlapping the easy-and-fast conception with solution genuineness. For that the players' payoff functions are going to be sampled finitely, with the pre-assigned constant sampling step in each of $\sum_{i=1}^{N} M_i$ dimensions, regarding special requirements to the sampling, which are to ensure sufficient accurateness for practice experience. Next task is to reconfigure the sets of the players' pure strategies in the finite game for its total simplification, which will allow to get rid off dimensionalities and to have the single dimension at each player. Eventually, every player's finite equilibrium strategy in the finite game as the approximation of the initial infinite game (4) mustn't be too dependent upon the sampling constants. Hence requirements to every strategy support should be stated so that it would be independent upon the sampling constants within some tolerable dependence.

4 Sampling the players' payoff functions

Let $S_m^{\langle n \rangle}$ be the sampling constant in m-th dimension of hypercube (1). The sampling constant $S_m^{\langle n \rangle}$ is number of equal-measure intervals between the selected points in m-th dimension of hypercube (1), and $\frac{1}{S_m^{\langle n \rangle}}$ is the constant sampling step in $(m + \sum_{i=1}^{n-1} M_i)$-th dimension of hypercube (3) $\forall m = \overline{1, M_n}$ and $\forall n = \overline{1, N}$. Of course, endpoints of every unit segment of hypercube (1) are included into the sampling necessarily. Then $S_m^{\langle n \rangle} \in \mathbb{N}$ for tolerating the utmost case of sampling. Thus m-th dimension of hypercube (1) for n-th player as the unit segment $[0;\, 1]$ is sampled with the constant sampling step $\frac{1}{S_m^{\langle n \rangle}}$. Instead of this segment, whose values constitute m-th component of the n-th player's pure strategy $\mathbf{X}_n$, there is the set of points

$$L_m^{\langle n \rangle}\left(S_m^{\langle n \rangle}\right) = \left\{x_{nm}^{\langle s_m \rangle}\right\}_{s_m=1}^{S_m^{\langle n \rangle}+1} \text{ by } x_{nm}^{\langle s_m \rangle} = \frac{s_m - 1}{S_m^{\langle n \rangle}} \;\; \forall m = \overline{1, M_n} \text{ and } \forall n = \overline{1, N}.$$

Following this, the n-th player's pure strategies set becomes

$$L^{\langle n \rangle}\left(\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right) = \bigtimes_{m=1}^{M_n} L_m^{\langle n \rangle}\left(S_m^{\langle n \rangle}\right) = \bigtimes_{m=1}^{M_n} \left\{\left\{x_{nm}^{\langle s_m \rangle}\right\}_{s_m=1}^{S_m^{\langle n \rangle}+1}\right\} \subset H_n. \tag{5}$$

And so the n-th player's payoff function is sampled, fixing its payoff values

$$K_n(\mathbf{X}) \text{ by } \mathbf{X}_n \in L^{\langle n \rangle}\left(\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right)$$

and letting the game (4) be converted into the finite one

$$\left\langle \left\{L^{\langle n \rangle}\left(\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right)\right\}_{n=1}^{N}, \left\{K_n(\mathbf{X})\right\}_{n=1}^{N} \right\rangle \text{ by } \mathbf{X}_n \in L^{\langle n \rangle}\left(\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right). \tag{6}$$

Note that the finite game (6) is defined on hypercubic lattice

$$L\left(\left\{\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right\}_{n=1}^{N}\right) = \bigtimes_{n=1}^{N} L^{\langle n \rangle}\left(\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right) = \bigtimes_{n=1}^{N} \bigtimes_{m=1}^{M_n} \left\{\left\{x_{nm}^{\langle s_m \rangle}\right\}_{s_m=1}^{S_m^{\langle n \rangle}+1}\right\} \subset H \tag{7}$$

now.

Clearly that constants

$$\left\{\left\{S_m^{\langle n \rangle}\right\}_{m=1}^{M_n}\right\}_{n=1}^{N} \tag{8}$$

shouldn't be assigned arbitrarily. They must be such that specificities of the players' payoff functions $\{K_n(\mathbf{X})\}_{n=1}^{N}$ would be preserved. These specificities consist mainly in local extremums and gradient over hypersurfaces $\{K_n(\mathbf{X})\}_{n=1}^{N}$. Supposing that these hypersurfaces are differentiable with respect to any of variables $\{\{x_{nm}\}_{m=1}^{M}\}_{n=1}^{N}$ and there exist mixed derivatives of each of functions $\{K_n(\mathbf{X})\}_{n=1}^{N}$ by any combination of variables $\{\{x_{nm}\}_{m=1}^{M}\}_{n=1}^{N}$ in any situation (2), where every variable is included no more than just once, there are the following requirements. Formally, $\forall s_m = \overline{1, S_m^{\langle n \rangle}}$ there ought to be

$$\frac{\partial^{\sum_{i=1}^{N} M_i} K_n(\mathbf{X})}{\partial x_{11} \partial x_{12} \ldots \partial x_{1M_1} \partial x_{21} \partial x_{22} \ldots \partial x_{2M_2} \ldots \partial x_{N1} \partial x_{N2} \ldots \partial x_{NM_N}} \geqslant 0 \text{ or}$$

$$\frac{\partial^{\sum_{i=1}^{N} M_i} K_n(\mathbf{X})}{\partial x_{11} \partial x_{12} \ldots \partial x_{1M_1} \partial x_{21} \partial x_{22} \ldots \partial x_{2M_2} \ldots \partial x_{N1} \partial x_{N2} \ldots \partial x_{NM_N}} \leqslant 0$$

$$\forall x_{nm} \in \left[x_{nm}^{\langle s_m \rangle}; x_{nm}^{\langle s_m+1 \rangle} \right], \;\; \forall m = \overline{1, M_n}, \;\; \forall n = \overline{1, N}. \tag{9}$$

But requirements (9) are obviously satisfied only if there are no extremums or discontinuities on any of intervals

$$\left\{ \left\{ \left\{ \left(x_{nm}^{\langle s_m \rangle}; x_{nm}^{\langle s_m+1 \rangle} \right) \right\}_{s_m=1}^{S_m^{\langle n \rangle}} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}.$$

So they can hardly be satisfied, unless there are two players and minimum of hypercube (3) dimensions. Far more real requirements are that on every of segments

$$\left\{ \left\{ \left\{ \left[x_{nm}^{\langle s_m \rangle}; x_{nm}^{\langle s_m+1 \rangle} \right] \right\}_{s_m=1}^{S_m^{\langle n \rangle}} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}$$

fluctuations of the players' payoff functions would be no greater than some $\alpha > 0$. Properly, $\forall s_m = \overline{1, S_m^{\langle n \rangle}}$ there ought to be

$$\left| \frac{\partial^{\sum_{i=1}^{N} M_i} K_n(\mathbf{X})}{\partial x_{11} \partial x_{12} \ldots \partial x_{1M_1} \partial x_{21} \partial x_{22} \ldots \partial x_{2M_2} \ldots \partial x_{N1} \partial x_{N2} \ldots \partial x_{NM_N}} \right| \leqslant \alpha$$

$$\forall x_{nm} \in \left[x_{nm}^{\langle s_m \rangle}; x_{nm}^{\langle s_m+1 \rangle} \right], \;\; \forall m = \overline{1, M_n}, \;\; \forall n = \overline{1, N}. \tag{10}$$

Based on practical reasonings,

$$\alpha \leqslant \beta \cdot \left(\max_{n \in \{\overline{1,N}\}} \max_{\mathbf{X} \in H} K_n(\mathbf{X}) - \min_{n \in \{\overline{1,N}\}} \min_{\mathbf{X} \in H} K_n(\mathbf{X}) \right)$$

by, say, $\beta = 0.05$, $\beta = 0.01$, $\beta = 0.005$ or $\beta = 0.001$, what shall be sufficiently accurate for practice experience. However, the parameter α may be taken lesser to have the game approximate solution consistent enough, what is going to spoken about below.

5 Reshaping multidimensional matrices of players' payoff values

In the finite game (6), the n-th player's payoff function, defined on hypercubic lattice (7), is represented as $(\sum_{i=1}^{N} M_i)$-dimensional matrix

$$\mathbf{P}_n(S_0) = \left[p_J^{\langle n \rangle}(S_0)\right]_{\mathscr{F}_0} \tag{11}$$

with denotation

$$S_0 = \left\{ \left\{ S_m^{\langle n \rangle} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N} = \left\{ S_{0,n} \right\}_{n=1}^{N} \tag{12}$$

of the format

$$\mathscr{F}_0 = \mathop{\times}_{n=1}^{N} \mathop{\times}_{m=1}^{M_n} \left(S_m^{\langle n \rangle} + 1 \right),$$

whose $(\sum_{i=1}^{N} M_i)$-position indices

$$\begin{gathered} J = \{j_d\}_{d=1}^{\sum_{i=1}^{N} M_i} \;\; \text{by} \;\; j_k \in \left\{ \overline{1, S_m^{\langle r \rangle} + 1} \right\} \;\; \text{at} \;\; k = m + \sum_{i=1}^{r-1} M_i \\ \forall m = \overline{1, M_r} \;\; \text{and} \;\; \forall r = \overline{1, N} \end{gathered} \tag{13}$$

determine the matrix element

$$p_J^{\langle n \rangle}(S_0) = K_n(\mathbf{X}) \;\; \text{by} \;\; x_{rm} = \frac{j_k - 1}{S_m^{\langle r \rangle}}. \tag{14}$$

In computations, there is a known rule telling that manipulating with a many single-dimensional objects is more convenient than manipulating with single multidimensional object [Hoffbeck, Sarwar, and Rix (2001); Rahman and Valdman (2013); Trapani, Kipouros, and Savill (2012)]. Practically it is explained with that the greater supplementary dimensions of a matrix the longer computations might be. This computational retardation is easy exampled in Matlab environment. Suppose that a 12-dimensional $\mathop{\times}_{d=1}^{12} 4$-matrix represents the player's payoff values in 3-person game (each player's pure strategy is of four dimensions with four points in every dimension). While operating on AMD Athlon II X2 250u Processor with 2 GB RAM within 64-bit Windows 7, 1400 Matlab operations of summing and extracting mean, finding minimal and maximal elements of three such matrices take about 3756 seconds, whereas the same takes about 2919 seconds over three six-dimensional $\mathop{\times}_{d=1}^{6} 16$-matrices, reshaped before. Moreover, reshaping these $\mathop{\times}_{d=1}^{12} 4$-matrices into three-dimensional $\bigtimes_{d=1}^{3}$ 256-matrices reduces the operation time down to 2691 seconds. Therefore matrices $\{\mathbf{P}_n(S_0)\}_{n=1}^{N}$ should be reshaped to reduce number of their dimensions ultimately. The minimal number of dimensions, apparently, is number of players. Here maintenance of one-to-one indexing is provided with the next theorem.

Theorem 1. There is a one-to-one indexing map of $\mathscr{F}_0$-matrix (11) into matrix of the format

$$\mathscr{L}_0 = \bigtimes_{r=1}^{N} \prod_{m=1}^{M_r} \left(S_m^{\langle r \rangle} + 1 \right).$$

This map is reversible.

Proof. Let $\mathscr{F}_0$-matrix (11) be reshaped into $\mathscr{L}_0$-matrix

$$\mathbf{G}_n(S_0) = \left[g_I^{\langle n \rangle}(S_0) \right]_{\mathscr{L}_0} \tag{15}$$

whose elements

$$g_I^{\langle n \rangle}(S_0) = p_J^{\langle n \rangle}(S_0) \tag{16}$$

have N-position indices, gathered within the set

$$I = \{u_r\}_{r=1}^{N} \text{ by } u_r = \sum_{m=1}^{M_r} \left(\prod_{w=1}^{m-1} \left(S_{M_r-w+1}^{\langle r \rangle} + 1 \right) \right) \cdot \left(j_{M_0-m+1} - \operatorname{sign}(m-1) \right)$$
$$\text{at } M_0 = \sum_{i=1}^{r} M_i \ \ \forall r = \overline{1, N}. \tag{17}$$

Having denoted

$$Q_n(S_{0,n}) = \prod_{m=1}^{M_n} \left(S_m^{\langle n \rangle} + 1 \right),$$

the convolution mapping (17) shows that $u_r = \overline{1, Q_r(S_{0,r})}$. Reversely, let the function $\psi(a, b)$ by $b \neq 0$ round the fraction $\frac{a}{b}$ to the nearest integer towards zero. And put another function

$$\rho(a, b) = a - b \cdot \psi(a, b). \tag{18}$$

By the function (18), the last index in indicating the n-th player's aggregate index u_n for matrix (11) is

$$j_{M_0} = \rho\left(u_n, S_{M_n}^{\langle n\rangle} + 1\right) + \left(S_{M_n}^{\langle n\rangle} + 1\right)\left(1 - \operatorname{sign}\left[\rho\left(u_n, S_{M_n}^{\langle n\rangle} + 1\right)\right]\right) \text{ at } M_0 = \sum_{i=1}^{n} M_i. \tag{19}$$

The rest $M_n - 1$ indices are

$$j_{M_0-m} = 1 + \rho\left(\frac{u_n - j_{M_0} - \sum\limits_{w=1}^{m-1}\left(\prod\limits_{w_1=1}^{w}\left(S_{M_n-w_1+1}^{\langle n\rangle}+1\right)\right)\cdot\left(j_{M_0-w}-1\right)}{\prod\limits_{w=1}^{m}\left(S_{M_n-w+1}^{\langle n\rangle}+1\right)}, S_{M_n-m}^{\langle n\rangle}+1\right)$$
$$\forall m = \overline{1, M_n - 1}. \tag{20}$$

Thus convolving statement (17) along with expansion (19) and (20) give the one-to-one reversible map of set $J = \{j_d\}_{d=1}^{\sum_{i=1}^{N} M_i}$ into set $I = \{u_r\}_{r=1}^{N}$. The theorem has been proved.

Theorem 1 establishes correspondence of subset of indices

$$\{j_k\}_{k=1+\sum_{i=1}^{n-1} M_i}^{\sum_{i=1}^{n} M_i} \subset J \tag{21}$$

to a pure strategy of n-th player and backwards, $n = \overline{1, N}$. This simplifies game (6) formalism ultimately, mapping the game

$$\left\langle \left\{ L^{\langle n\rangle}\left(\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}\right)\right\}_{n=1}^{N}, \left\{\mathbf{P}_n\left(S_0\right)\right\}_{n=1}^{N}\right\rangle \tag{22}$$

into

$$\left\langle \left\{\left\{ z_{u_n}^{\langle n\rangle}\left(S_{0,n}\right)\right\}_{u_n=1}^{Q_n(S_{0,n})}\right\}_{n=1}^{N}, \left\{\mathbf{G}_n\left(S_0\right)\right\}_{n=1}^{N}\right\rangle, \tag{23}$$

where the n-th player's pure strategy $z_{u_n}^{\langle n\rangle}\left(S_{0,n}\right)$ corresponds to its strategy $\mathbf{X}_n$ in the initial game (4), whose components are

$$\left\{x_{nm} = \frac{j_k - 1}{S_m^{\langle n\rangle}}\right\}_{m=1}^{M_n} \text{ at } k = m + \sum_{i=1}^{n-1} M_i \text{ by } n = \overline{1, N}$$

due to subset (21) and index u_n correspondence.

6 Consistency of equilibrium strategy support, approximating the unknown genuine equilibrium strategy

Without mentioning a method of solving the finite game (23), there is supposition of that the game (23) solution

$$\left\{ \left\{ p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{0,n} \right) \right) \right\}_{u_n=1}^{Q_n(S_{0,n})} \right\}_{n=1}^{N} \tag{24}$$

is known by the probability $p_n^*(z_{u_n}^{\langle n \rangle}(S_{0,n}))$ of applying the pure strategy $z_{u_n}^{\langle n \rangle}(S_{0,n})$ in an equilibrium strategy of n-th player

$$\left\{ p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{0,n} \right) \right) \right\}_{u_n=1}^{Q_n(S_{0,n})} . \tag{25}$$

And may the support of the n-th player's strategy (25) be the set

$$\operatorname{supp} \left\{ p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{0,n} \right) \right) \right\}_{u_n=1}^{Q_n(S_{0,n})} = Z_n^* \left(S_0 \right) = \left\{ z_{u_n^*}^{\langle n \rangle} \left(S_{0,n} \right) \right\}_{u_n^* \in U_n^*(S_0)} \tag{26}$$

with its cardinality $|U_n^*(S_0)|$, whence

$$p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{0,n} \right) \right) > 0 \;\; \forall u_n \in U_n^* \left(S_0 \right) \tag{27}$$

and

$$p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{0,n} \right) \right) = 0 \;\; \forall u_n \notin U_n^* \left(S_0 \right).$$

For seeing whether the strategy support (26) is independent upon the sampling constants (8) within some tolerable dependence, consider δ-neighboring to (23) game

$$\left\langle \left\{ \left\{ z_{u_n}^{\langle n \rangle} \left(S_{\delta,n} \right) \right\}_{u_n=1}^{Q_n\left(S_{\delta,n}\right)} \right\}_{n=1}^{N}, \left\{ \mathbf{G}_n \left(S_\delta \right) \right\}_{n=1}^{N} \right\rangle$$

with denotations

$$S_\delta = \left\{ \left\{ S_m^{\langle n \rangle} + \delta \right\}_{m=1}^{M_n} \right\}_{n=1}^{N} = \left\{ S_{\delta,n} \right\}_{n=1}^{N} \;\text{ by }\; \delta \in \mathbb{Z},$$

$$Q_n \left(S_{\delta,n} \right) = \prod_{m=1}^{M_n} \left(S_m^{\langle n \rangle} + \delta + 1 \right),$$

$$\mathbf{G}_n \left(S_\delta \right) = \left[g_I^{\langle n \rangle} \left(S_\delta \right) \right]_{\mathscr{L}_\delta},$$

$$\mathscr{L}_{\delta} = \bigtimes_{r=1}^{N} \prod_{m=1}^{M_r} \left(S_m^{\langle r \rangle} + \delta + 1 \right),$$

$$\left\{ \left\{ p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{\delta,n} \right) \right) \right\}_{u_n=1}^{Q_n\left(S_{\delta,n}\right)} \right\}_{n=1}^{N}, \tag{28}$$

$$\operatorname{supp} \left\{ p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{\delta,n} \right) \right) \right\}_{u_n=1}^{Q_n\left(S_{\delta,n}\right)} = Z_n^* \left(S_{\delta} \right) = \left\{ z_{u_n^*}^{\langle n \rangle} \left(S_{\delta,n} \right) \right\}_{u_n^* \in U_n^* \left(S_{\delta} \right)}, \tag{29}$$

where $\left| U_n^* \left(S_{\delta} \right) \right|$ is cardinality of the set (29), and $\left(\sum_{i=1}^{N} M_i \right)$-position indices (13) are mapped into N-position indices (17) by putting $S_m^{\langle n \rangle} \equiv S_m^{\langle n \rangle} + \delta$ into (13), (14), (16), (17), (19), (20).

An aggregate feature of situation (28) or supports $\left\{ Z_n^* \left(S_{\delta} \right) \right\}_{n=1}^{N}$ is the i-th player's payoff, being taken in this situation:

$$v_i^* \left(S_{\delta} \right) = \sum_{\substack{I = \left\{ u_l \right\}_{l=1}^{N} \\ u_l = \overline{1, Q_l \left(S_{\delta,l} \right)}}} \left(g_I^{\langle i \rangle} \left(S_{\delta} \right) \cdot \prod_{n=1}^{N} p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{\delta,n} \right) \right) \right)$$

$$= \sum_{I^* \left(S_{\delta} \right) = \left\{ u_l^* : u_l^* \in U_l^* \left(S_{\delta} \right) \right\}_{l=1}^{N}} \left(g_{I^* \left(S_{\delta} \right)}^{\langle i \rangle} \left(S_{\delta} \right) \cdot \prod_{n=1}^{N} p_n^* \left(z_{u_n^*}^{\langle n \rangle} \left(S_{\delta,n} \right) \right) \right), \quad i = \overline{1, N}.$$

Apparently, there can be selected such constants (8), for which at least $\exists n_0 \in \left\{ \overline{1, N} \right\}$ such that payoffs $v_{n_0}^* \left(S_0 \right)$ and $v_{n_0}^* \left(S_1 \right)$ will be significantly different. In other words, decreasing sampling steps minimally may give payoffs $\left\{ v_n^* \left(S_1 \right) \right\}_{n=1}^{N}$ as if they do not relate to payoffs $\left\{ v_n^* \left(S_0 \right) \right\}_{n=1}^{N}$. Similarly, decreasing sampling steps minimally may give an equilibrium situation

$$\left\{ \left\{ p_n^* \left(z_{u_n}^{\langle n \rangle} \left(S_{1,n} \right) \right) \right\}_{u_n=1}^{Q_n\left(S_{1,n}\right)} \right\}_{n=1}^{N}, \tag{30}$$

whose configuration is hardly comparable to the corresponding configuration of situation (24). Therefore, approximating the unknown genuine equilibrium strategy requires two items. Firstly, payoffs $\left\{ v_n^* \left(S_0 \right) \right\}_{n=1}^{N}$ and $\left\{ v_n^* \left(S_1 \right) \right\}_{n=1}^{N}$, taken in situations (24) and (30), should be sufficiently close in $\mathbb{R}$-metric. Secondly, situations (24) and (30) themselves should be sufficiently close uniformly and in $\left\{ \mathbb{L}_2 \left(H_n \right) \right\}_{n=1}^{N}$-metrics.

Of course, the spoken sufficient closeness is relative, meaning that the attribute value (every player's payoff and its strategy) differentiates no greater as the numbers of the set (12) increase minimally (in comparison to that when they decrease minimally). Relativity is unavoidable because neither genuine payoffs $\left\{ v_n^{**} \right\}_{n=1}^{N}$ in the game (4), nor limits

$$\lim_{\delta\to\infty} v_n^*(S_\delta) \quad \forall n=\overline{1,N} \tag{31}$$

are known, where v_n^{**} is the n-th player's payoff in the equilibrium situation, being approximated with (28) by $\delta \in \mathbb{Z}$. The same concerns δ-neighboring situation (28), whose convergence by $\delta \to \infty$ to the genuine equilibrium situation in the game (4) is not proved.

Sufficient closeness of the players' payoffs is that

$$|v_n^*(S_0) - v_n^*(S_1)| \leqslant |v_n^*(S_{-1}) - v_n^*(S_0)| \quad \forall n=\overline{1,N}. \tag{32}$$

Sufficient closeness of situations, giving those payoffs in (32), needs consideration of the player's strategy finite support as a hypersurface. Henceforward in the finite game (23), let n-th player have a piecewise linear hypersurface $\sigma_n(u_n, S_0)$, whose vertices are in points

$$\left\{\left\{\left[\frac{j_k-1}{S_m^{\langle n\rangle}}\right]_{1\times M_n} \in \mathbb{R}^{M_n} : k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1,M_n}\right\},\, p_n^*\left(z_{u_n}^{\langle n\rangle}(S_{0,n})\right)\right\}$$

in the space $\mathbb{R}^{M_n+1}$. The n-th player's strategy support (26) scores up $|U_n^*(S_0)|$ vertices of the hypersurface $\sigma_n(u_n, S_0)$ with (27), wherein they correspond to point with denotation

$$\mathbf{X}_n^{\langle q\rangle}(S_0) = \left[x_{nm}^{\langle q\rangle}(S_0)\right]_{1\times M_n} = \left[\frac{j_k^{\langle q\rangle}(S_0)-1}{S_m^{\langle n\rangle}}\right]_{1\times M_n} \in H_n \text{ by}$$

$$k=m+\sum_{i=1}^{n-1} M_i \text{ and } m=\overline{1,M_n}$$

for $q=\overline{1,|U_n^*(S_0)|}$ while every index $u_n^* \in U_n^*(S_0)$ is expanded into M_n indices by Theorem 1. Properly speaking, the set

$$\left\{\mathbf{X}_n^{\langle q\rangle}(S_0)\right\}_{q=1}^{|U_n^*(S_0)|} \subset H_n \tag{33}$$

is the n-th player's equilibrium strategy support in the game (6). And let the set (33) be sorted into the set

$$\left\{\bar{\mathbf{X}}_n^{\langle q\rangle}(S_0)\right\}_{q=1}^{|U_n^*(S_0)|} = \left\{\left[\frac{\bar{j}_k^{\langle q\rangle}(S_0)-1}{S_m^{\langle n\rangle}}\right]_{1\times M_n}\right\}_{q=1}^{|U_n^*(S_0)|} = \left\{\mathbf{X}_n^{\langle q\rangle}(S_0)\right\}_{q=1}^{|U_n^*(S_0)|}$$

$$\text{by } k = m + \sum_{i=1}^{n-1} M_i \text{ and } m = \overline{1, M_n} \tag{34}$$

so that the value

$$\min_{q_1 \in \left\{\overline{q+1, |U_n^*(S_0)|}\right\}} \sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1,M_n}} \left(\frac{\bar{j}_k^{\langle q \rangle}(S_0) - \bar{j}_k^{\langle q_1 \rangle}(S_0)}{S_m^{\langle n \rangle}} \right)^2} \tag{35}$$

is reached at $q_1 = q+1$ for each $q = \overline{1, |U_n^*(S_0)| - 1}$ and $n = \overline{1, N}$. This is being made for evaluating sufficient closeness of finite strategies, when the sampling constants (8) vary minimally. Partially, this sufficient closeness is that

$$\max_{H_n} |\sigma_n(u_n, S_0) - \sigma_n(u_n, S_1)| \leqslant \max_{H_n} |\sigma_n(u_n, S_{-1}) - \sigma_n(u_n, S_0)| \tag{36}$$

and

$$\|\sigma_n(u_n, S_0) - \sigma_n(u_n, S_1)\| \leqslant \|\sigma_n(u_n, S_{-1}) - \sigma_n(u_n, S_0)\| \text{ in } \mathbb{L}_2(H_n). \tag{37}$$

Fully, the upper support density shall also not decrease by the decreased sampling steps minimally:

$$\max_{q \in \left\{\overline{1, |U_n^*(S_1)|-1}\right\}} \sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1,M_n}} \left(\frac{\bar{j}_k^{\langle q \rangle}(S_1) - \bar{j}_k^{\langle q+1 \rangle}(S_1)}{S_m^{\langle n \rangle}+1} \right)^2} \leqslant$$
$$\max_{q \in \left\{\overline{1, |U_n^*(S_0)|-1}\right\}} \sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1,M_n}} \left(\frac{\bar{j}_k^{\langle q \rangle}(S_0) - \bar{j}_k^{\langle q+1 \rangle}(S_0)}{S_m^{\langle n \rangle}} \right)^2} \quad \forall n = \overline{1, N}. \tag{38}$$

The definition below, engaged sufficient closeness in every player's payoff and its strategy, is to see whether (24) is worth to count it the approximate solution of the game (4).

Definition 1. The solution (24) of the game (23) is called weakly consistent for being the approximate solution of the game (4) if the inequalities

$$|U_n^*(S_1)| \geqslant |U_n^*(S_0)| \quad \forall n = \overline{1, N} \tag{39}$$

are true along with (32) and (36)–(38). Every strategy and its support in weakly consistent solution are called weakly consistent.

Inequalities (39) express a natural requirement that cardinality of every player's strategy support shall not decrease as the sampling steps by constants (8) decrease minimally. Weakness of consistency has been inserted inasmuch as requirements of the non-decreasing upper support density and support cardinality in (38) and (39) can be strengthened.

Definition 2. The weakly consistent solution (24) of the game (23) is called consistent for being the approximate solution of the game (4) if the inequalities

$$\left|U_n^*\left(S_0\right)\right| \geqslant \left|U_n^*\left(S_{-1}\right)\right| \quad \forall n=\overline{1, N} \tag{40}$$

and

$$\max_{q \in \left\{\overline{1, \left|U_n^*\left(S_0\right)\right|-1}\right\}} \sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1, M_n}} \left(\frac{\bar{j}_k^{\langle q\rangle}\left(S_0\right)-\bar{j}_k^{\langle q+1\rangle}\left(S_0\right)}{S_m^{\langle n\rangle}}\right)^2} \leqslant \tag{41}$$

$$\max_{q \in \left\{\overline{1, \left|U_n^*\left(S_{-1}\right)\right|-1}\right\}} \sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1, M_n}} \left(\frac{\bar{j}_k^{\langle q\rangle}\left(S_{-1}\right)-\bar{j}_k^{\langle q+1\rangle}\left(S_{-1}\right)}{S_m^{\langle n\rangle}-1}\right)^2} \quad \forall n=\overline{1, N}$$

are true. Every strategy and its support in consistent solution are called consistent. Well, consistency or, accurately, S_0-consistency subtends five stands, implicating the sets $\{S_{-1}, S_0, S_1\}$. All issues from the player's support: the support configuration, generating the payoffs in the equilibrium situation, the support cardinality, and the upper support density. Appositely, the support density is treated absolute, using Euclidean distance between points of the support. This has been acted for distinguishing the upper support density of the player's completely mixed strategy as the sampling constants (8) vary. Hence the assertion below forces itself.

Theorem 2. If weakly consistent strategy is completely mixed then it is consistent.

Proof. Let weakly consistent equilibrium strategy (25) of n-th player be completely mixed. Then

$$\left|U_n^*\left(S_0\right)\right| = Q_n\left(S_{0,n}\right).$$

Easily noting that

$$Q_n\left(S_{0,n}\right)=\prod_{m=1}^{M_n}\left(S_m^{\langle n\rangle}+1\right)>\prod_{m=1}^{M_n} S_m^{\langle n\rangle}=Q_n\left(S_{-1,n}\right)$$

and

$$Q_n\left(S_{-1,n}\right) \geqslant \left|U_n^*\left(S_{-1}\right)\right|,$$

have

$$\left|U_n^*\left(S_0\right)\right| = Q_n\left(S_{0,n}\right) > Q_n\left(S_{-1,n}\right) \geqslant \left|U_n^*\left(S_{-1}\right)\right|, \tag{42}$$

what confirms the inequality (40). Because strategy (25) is completely mixed, its support is the set (5). Consequently, indices

$$\left\{\left\{\bar{j}_k^{\langle q\rangle}\left(S_0\right)\right\}_{k\in B}\right\}_{q=1}^{\left|U_n^*\left(S_0\right)\right|} \text{ at } B = \left\{m + \sum_{i=1}^{n-1} M_i\right\}_{m=1}^{M_n} \quad \forall n = \overline{1, N}$$

with their values

$$\left\{\left\{\bar{j}_k^{\langle q\rangle}\left(S_0\right) \in \left\{\overline{1,\, S_m^{\langle n\rangle}+1}\right\}\right\}_{k\in B}\right\}_{q=1}^{\left|U_n^*\left(S_0\right)\right|}$$

are such that there is the single $k_0 \in B$ such that

$$\bar{j}_{k_0}^{\langle q+1\rangle}\left(S_0\right) = \bar{j}_{k_0}^{\langle q\rangle}\left(S_0\right) + 1$$

and

$$\bar{j}_k^{\langle q+1\rangle}\left(S_0\right) = \bar{j}_k^{\langle q\rangle}\left(S_0\right) \quad \forall k \in B \backslash \left\{k_0\right\}$$

for all $q = \overline{1,\, \left|U_n^*\left(S_0\right)\right| - 1}$. So that

$$\sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1,M_n}} \left(\frac{\bar{j}_k^{\langle q\rangle}\left(S_0\right) - \bar{j}_k^{\langle q+1\rangle}\left(S_0\right)}{S_m^{\langle n\rangle}}\right)^2} = \frac{1}{S_{m_0}^{\langle n\rangle}}$$

$$\text{by } m_0 = k_0 - \sum_{i=1}^{n-1} M_i \quad \forall q = \overline{1,\, \left|U_n^*\left(S_0\right)\right| - 1} \tag{43}$$

and, subsequent upon (42),

$$\begin{aligned} &\max_{q\in\left\{\overline{1,\left|U_n^*\left(S_0\right)\right|-1}\right\}} \sqrt{\sum_{k=m+\sum_{i=1}^{n-1} M_i,\, m=\overline{1,M_n}} \left(\frac{\bar{j}_k^{\langle q\rangle}\left(S_0\right) - \bar{j}_k^{\langle q+1\rangle}\left(S_0\right)}{S_m^{\langle n\rangle}}\right)^2} = \\ &\max\left\{\frac{1}{S_m^{\langle n\rangle}}\right\}_{m=1}^{M_n} = \frac{1}{\min\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}}. \end{aligned} \tag{44}$$

Value (43) is clear to be minimal upper support density. According to this,

$$\max_{q\in\left\{\overline{1,\left|U_n^*(S_{-1})\right|-1}\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\, m=\overline{1,M_n}}\left(\frac{\bar{j}_k^{\langle q\rangle}\left(S_{-1}\right)-\bar{j}_k^{\langle q+1\rangle}\left(S_{-1}\right)}{S_m^{\langle n\rangle}-1}\right)^2}\geqslant$$

$$\max\left\{\frac{1}{S_m^{\langle n\rangle}-1}\right\}_{m=1}^{M_n}=\frac{1}{\min\left\{S_m^{\langle n\rangle}-1\right\}_{m=1}^{M_n}}.$$

Once again easily noting that

$$\max\left\{\frac{1}{S_m^{\langle n\rangle}}\right\}_{m=1}^{M_n}=\frac{1}{\min\left\{S_m^{\langle n\rangle}\right\}_{m=1}^{M_n}}<\frac{1}{\min\left\{S_m^{\langle n\rangle}-1\right\}_{m=1}^{M_n}}=\max\left\{\frac{1}{S_m^{\langle n\rangle}-1}\right\}_{m=1}^{M_n} \tag{45}$$

have the inequality (41) confirmed. The theorem has been proved.

For the completely mixed strategy, inequalities (40) and (41) hold true strictly. Appositely, the case when inequalities (32) and (36)–(39) hold true strictly might have been called strict weak consistency, and strict consistency would have been on strict inequalities (32) and (36)–(41). That could be used in proving some limit theorems, but now questions of consistency computational approach are of higher importance.

Theorem 3. If the game

$$\left\langle\left\{\left\{z_{u_n}^{\langle n\rangle}\left(S_{1,n}\right)\right\}_{u_n=1}^{Q_n(S_{1,n})}\right\}_{n=1}^{N},\left\{\mathbf{G}_n\left(S_1\right)\right\}_{n=1}^{N}\right\rangle \tag{46}$$

has a completely mixed equilibrium situation, then for checking weak S_0-consistency of the same equilibrium type situation it is sufficient to check inequalities (32), (36), (37).

Proof. Apparently,

$$\left|U_n^*\left(S_1\right)\right|=Q_n\left(S_{1,n}\right)>Q_n\left(S_{0,n}\right)\geqslant\left|U_n^*\left(S_0\right)\right|\quad\forall n=\overline{1,N}$$

and

$$\max_{q\in\left\{\overline{1,\left|U_n^*(S_1)\right|-1}\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\, m=\overline{1,M_n}}\left(\frac{\bar{j}_k^{\langle q\rangle}\left(S_1\right)-\bar{j}_k^{\langle q+1\rangle}\left(S_1\right)}{S_m^{\langle n\rangle}+1}\right)^2}=$$

$$\max\left\{\frac{1}{S_m^{\langle n\rangle}+1}\right\}_{m=1}^{M_n}=\frac{1}{\min\left\{S_m^{\langle n\rangle}+1\right\}_{m=1}^{M_n}}<$$

$$\max_{q\in\left\{\overline{1,|U_n^*(S_0)|-1}\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\,m=\overline{1,M_n}}\left(\frac{\bar{j}_k^{\langle q\rangle}(S_0)-\bar{j}_k^{\langle q+1\rangle}(S_0)}{S_m^{\langle n\rangle}}\right)^2}\quad\forall n=\overline{1,N}$$

with applying (42) and (44) and (45) to the completely mixed strategies. So, inequalities (38) and (39) hold true, and there remain the inequalities (32), (36), (37) to be checked whether the game (23) solution (24) is weakly S_0-consistent. The theorem has been proved.

The proved assertions help either in reducing computations on consistency or rejecting the non-consistent solutions faster. Being valid on completely mixed strategies, they operate the support cardinality and upper support density. The investigator is brought to control inequalities (32), (36), (37), though.

7 Discussion and conclusive remarks

This article concerns conversion of the unit-hypercubic infinite noncooperative game into a finite game. Validity of the conversion is grounded on sampling the players' payoff functions regularly, whereupon the obtained finite game solution is checked for its consistency. The equilibrium type is not specified nevertheless. It may be as Nash equilibrium type, as well as a lot of the refined or modified principles of optimality, allowing to smooth differences in utility and equity: strong Nash equilibrium [Suh (2001); Tian (2000)], Pareto equilibrium [Gąsior and Drwal (2013); Vorob'yov (1984,1985); Scalzo (2010); Zhu, Liu, Wang, and Yu (2004); Li, Luo, and Sun (2011); Trapani, Kipouros, and Savill (2012)], perfect Bayesian equilibrium [Fudenberg and Tirole (1991); Battigalli (1996)], Mertens-stable equilibrium [Kohlberg and Mertens (1986)], Markov perfect equilibrium [Castro and Brandão (2000); Haller and Lagunoff (2010)], etc.

The proposed approximation of the infinite noncooperative game allows to solve the finite game easier thanking to that every player's payoff sampled function is reshaped into the line array, whereas computations over the multidimensional matrix with minimally possible number of dimensions are faster. Theorem 1 guarantees that the reshaping is the one-to-one indexing map, which is reversible. Reversibility is necessary in restoring the finite game (22) solution from the solution (24) of its simplified analogue (23).

Also the proposed approximation of the infinite noncooperative game calls for consistency of equilibrium strategy support, approximating the unknown genuine equilibrium strategy. Weak consistency by Definition 1 signifies that difference of the players' payoffs and difference of the players' equilibrium strategies and cardinalities of their supports are non-decreasing. This property becomes stronger by consistency in Definition 2, which imparts relative independence to the approximate solution upon the sampling steps within their minimal neighborhood in each of dimensions of hypercube (3).

In checking weak S_0-consistency of the approximate solution, there are $5N$ inequalities (32) and (36)–(39) to be checked. The check consecution starts with checking the inequalities (39), needing only solution of 1-neighboring to (23) game. If they all are true then inequalities (32) are checked, where (-1)-neighboring to (23) game is solved. If inequalities (32) are true then the support sufficient closeness in inequalities (36) and (37) is checked. And it is efficient that the problem (35) for sorting points (33) in the set (34) be solved after the sufficient closeness of equilibrium strategies is verified. Eventually, inequalities (38) are checked. Checking S_0-consistency should always follow the fact that the solution is weakly consistent. It starts with inequalities (40). If they all are true then inequalities (41), related to sorting problems, are checked. Namely the stated consecutions are preferable, because the easiest requirements are checked before the more complicated ones in order to prevent needless huge computations.

Deficiently, existence of limits (31) and their convergence to the genuine players' payoffs protrudes non-proved. Poorly that the limits

$$\lim_{\delta \to \infty} \sigma_n (u_n, S_\delta) \quad \forall n = \overline{1, N}$$

existence and their convergence to the being approximated equilibrium strategies have been left non-proved as well. Besides, it is unclear whether consistent strategy support causes at least the weak consistency of the other strategy support. Say, could a player's strategy support be inconsistent while the rest of $N-1$ players have their supports (weakly) consistent? Or else: shall a player use its (weakly) consistent strategy while the rest of $N-1$ players have inconsistent supports? After all, the consistency paradigm can be regarded not only to equilibrium strategies but also to others (for instance, when a player decides to spring off an equilibrium situation).

In spite of everything, either conditions within Definition 1 or conditions within Definition 2 suggest a proper approximation of the infinite noncooperative game (4). The game (4) isomorphism to noncooperative games by measurable and bounded payoff functions, defined on compact subspaces in $\mathbb{R}^{\sum_{n=1}^{N} M_n}$, ensure dissemination of that approximation approach on compact games. Further to this, the stated method of converting the unit-hypercubic infinite noncooperative game into the finite game lets have an equilibrium solution to the conflict object, even when the game (4) is solved in ε-equilibrium situations or doesn't have solution at all. And every player, having the finite strategy support, will practice it freer unlike practicing on infiniteness or with continuous variates. Computing the factual solution stays for finite noncooperative game solvers [Osborne (2003); Vorob'yov (1984); Nisan, Roughgarden, Tardos, and Vazirani (2007); Kuhn (1961); Kontogiannis, Panagopoulou, and Spirakis (2009)], wherein the computation period is shortened due to the minimized number of dimensions.

The game approximation is going to be brought forward: for each player, there will not be necessarily equal-measure intervals between the selected points in every dimension of the player's pure strategies hypercube. This will let have irregular multidimensional hypercubic lattice instead of hypercube (1) wherewith to construct payoff matrices regarding straightforwardly any specificities, local extremums, and gradient over hypersurfaces $\{K_n(\mathbf{X})\}_{n=1}^{N}$, getting rid off requirements (9) and (10). Additionally, the stated S_0-consistency, implicated the sets $\{S_{-1}, S_0, S_1\}$ or δ-neighboring to (23) games by $\delta \in \{-1, 0, 1\}$, is going to be extended out to δ-neighborhoods, considering wider symmetric ranges of δ.

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