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The Optimal Radius of the Support of Radial Weights Used in Moving Least Squares Approximation

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Abstract: Owing to the meshless and local characteristics, moving least squares (MLS) methods have been used extensively to approximate the unknown function of partial differential equation initial boundary value problem. In this paper, based on matrix analysis, a sufficient and necessary condition for the existence of inverse of coefficient matrix used in MLS methods is developed firstly. Then in the light of approximate theory, a practical mathematics model is posed to obtain the optimal radius of support of radial weights used in MLS methods. As an example, while uniform distributed particles and the 4^{th} order spline weight function are adopted in MLS method in two dimension domain and two kinds of norms are used to measure error, optimal results for linear and quadratic basis are gained. Finally, the test data verify that the optimal values are correct. The research idea can be used in 3-dimension problems too.

keyword: MLS methods, Radius of support, Scaling, Sobolev norm, Mathematics model, Matrix analysis, Approximate theory.

1 Introduction

Comparing with the radial basis function interpolation approach, the moving least squares (MLS) method offers another kind of efficient scattered data approximation especially if the number of point is large and the data values contain noise. The MLS method is a variation on the classical least squares technique with the advantage allowing the nearest neighbors of the evaluation point x to influence the approximate value through a weight function with local compact support $w(x, x_j) : R^d \times R^d \to R^+$ where $x_j$is one of the given particles (nodes) in set $P_\Omega = \{x_j\}_{j=1}^n$in the bounded domain $\Omega \subset R^d$. That is for every point x we have to solve the following problem

$$\min_{s\in S}\left\{\sum_{j=1}^{n}\left[s(x_j)-f_j\right]^{2}w(x,x_j)\right\}, \tag{1}$$

where S is a finite-dimensional linear space and $f_j = f(x_j)$ is the collected data. Weight function $w(x,x_j)$ with the form $w_0\left(\left\|x-x_j\right\|_2/r_j\right)$ is generally used to simplify the form of weight function and help forward the independence of weight function on the dimension d of the domain Ω. As function $w_0(r)$has a compact support [0, 1], weight function $w(x,x_j)$has a disc support with center x_j and radius r_j. In this paper, we would like to take the radius as a constant r for simplicity.

The MLS approximation has its origin in the early paper [Lancaster and Salkauskas(1981)] with special cases going back to [McLain(1974),and Shepard(1968)], and some investigation about the approximation order is given in paper [Farwig(1986)]. Now MLS methods have emerged as the basis of numerous meshless (meshfree) approximation methods that being suggested as an alternative to the traditional finite element method in references [Atluri (2004),and Babuska, Banerjee and Osborn(2003), Liu, Han and Lu (2004)] and there referred. Especially, the generalized moving least squares methods is developed and successfully applied as a approximation methods to solve thin beam problem in paper [Atluri, Cho, Kim(1999)].

As we know, one of the crucial steps to solve partial different equations system is the approximation to the unknown field function appeared in the system, i.e. how to ensure trial function being an effective approximation. The influence factors existing in MLS method include which kind of weight function w_0 should be used and how much the size of compact support of the weight function r should be. Reference [Atluri (2004)] suggests to use 4^{th} order spline type weight function in order to give smoothness to the derivatives of the trail function. About the second factor there is no definite answer up to now as our known.

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In this paper, after a brief introduction of the MLS approximation in this section, a sufficient and necessary condition about the existence of the inverse of coefficient matrix of linear equations system used in MLS method is posed and proved when uniform distributed particle is exploited in section 2. The conclusions about the application in the case of linear and quadratic basis are specified in section 3. Then the model of optimal radius of the support of radial weight function is developed and solved in section 4 and 5 respectively. Some numerical tests about the optimal radius when linear bases and quadrics base being used are given in section 6. And the conclusions are shown in the end.

2 Sufficient and necessary

Let's assume that the finite dimension of linear space S used in formula (1) is expressed as

$$S = span\{p_1(x), p_2(x), \cdots, p_m(x)\} \quad ,$$

i.e. a series of linear independent functions $p_1(x), p_2(x), \cdots, p_m(x)$ defined on R^d are the basis of linear space S, and for any $s \in S$ there exist a group of coefficients $\{a_i\}_{i=1}^m \subset R$ such that $s = \sum_{i=1}^m a_i p_i(x)$. Then for any given point $x \in \Omega \subset R^d$, the moving least squares problem (1) can be writen as

Find $s^* = \sum_{i=1}^m a_i(x) p_i(x)$ such that

$$\begin{aligned} &\min_{s\in S} \left\{ \sum_{j=1}^n [s(x_j) - f_j]^2 w(x,x_j) \right\} \\ &= \min_{\substack{a_i \in R \\ 1\le i\le m}} \left\{ \sum_{j=1}^n \left[\sum_{i=1}^m a_i p_i(x_j) - f_j \right]^2 w(x,x_j) \right\} \\ &= \sum_{j=1}^n \left[\sum_{i=1}^m a_i(x) p_i(x_j) - f_j \right]^2 w(x,x_j). \end{aligned} \tag{2}$$

According to least squares principle, for any point $x \in \Omega \subset R^d$, the coefficients $\{a_i(x)\}_{i=1}^m \subset R$ of the solution function s^* should be the solution of the following linear equations system

$$\mathbf{A}(x)\mathbf{a}(x) = \mathbf{B}(x)\mathbf{u}, \tag{3}$$

where matrix

$$\begin{aligned} &\mathbf{A}(x) = \mathbf{P}^T\mathbf{W}(x)\mathbf{P}, \quad \mathbf{B}(x) = \mathbf{P}^T\mathbf{W}(x), \\ &\mathbf{W}(x) = diag\{w(x,x_1),\, w(x,x_2),\, \ldots, w(x,x_n)\}, \end{aligned} \tag{4}$$

$$\mathbf{P} = \begin{bmatrix} p_1(x_1)p_2(x_1)\ldots p_m(x_1) \\ p_1(x_2)p_2(x_2)\cdots p_m(x_2) \\ \ldots\ldots\ddots\ldots \\ p_1(x_n)p_2(x_n)\cdots p_m(x_n) \end{bmatrix}, \quad \mathbf{a}(x) = \begin{bmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_m(x) \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}. \tag{5}$$

In order to describe the solvability of the linear equations system (3) clearly, we need introduce the following definition as in reference [Zuo and Nie(2005)].

Definition 1 Assuming that we have functions series $\{\varphi_i(x)\}_{i=1}^m$ and particles set $X = \{x_j\}_{j=1}^n$, $\{\varphi_i(x)\}_{i=1}^m$ is said to be linear independent on the set X if equations $\sum_{i=1}^m c_i\varphi_i(x_j) = 0 \quad (1 \le j \le n)$ lead to $c_1 = c_2 = \cdots = c_m = 0$.

Remark 2 According to the definition, obviously the functions series $\{\varphi_i(x)\}_{i=1}^m$ is linear independent on the set X if and only if the rank of the matrix $\mathbf{\Phi}$ is m. The matrix $\mathbf{\Phi}$ is defined as

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_1(x_1)\varphi_2(x_1)\ldots\varphi_m(x_1) \\ \varphi_1(x_2)\varphi_2(x_2)\cdots\varphi_m(x_2) \\ \ldots\ldots\ddots\ldots \\ \varphi_1(x_n)\varphi_2(x_n)\cdots\varphi_m(x_n) \end{bmatrix}.$$

In other words, the column vectors of matrix Φ are linear independent. The geometric explanation is that there is a group of particles $\{x_{ij}\}_{j=1}^m \subset X$ such that they do not locate in any same curve expressed by a function in space $span\{\varphi_1(x), \varphi_2(x), \cdots, \varphi_m(x)\}$.

Now, let us come back to the MLS approximation. For a fixed $x \in \Omega$, suppose that there are number of k particles, $\{x_{i1}, x_{i2}, \cdots, x_{ik}\} \stackrel{\Delta}{=} P_x \subset P_\Omega$, which satisfy condition weight $w(x,x_l) > 0$ for any $x_l \in P_x$ in contrast with weight $w(x,x_l) = 0$ for $x_l \in P_\Omega \backslash P_x$. Reference [Atluri (2004)] calls P_x as the influence domain of point x based on given particles set P_Ω.

Take a permutation matrix $\mathbf{T}_x$ which can be used to realize the following transformation

$$\mathbf{T}_x \mathbf{W}(x) \mathbf{T}_x^T = \begin{bmatrix} \mathbf{W}_{11}(x) & \mathbf{O}_{12} \\ \mathbf{O}_{21} & \mathbf{O}_{22} \end{bmatrix}, \tag{6}$$

where matrix

$$\mathbf{W}_{11}(x) = diag\{w(x,x_{i1}), w(x,x_{i2}), \cdots, w(x,x_{ik})\},$$

and the sizes of the zero matrixes $\mathbf{O}_{12}$, $\mathbf{O}_{21}$, $\mathbf{O}_{22}$ are $k \times (n-k)$, $(n-k) \times k$, $(n-k) \times (n-k)$ respectively. Using the characteristic of permutation matrix, we have the alternative form of coefficient matrix of linear equations system (3) as follows

$$\begin{aligned} \mathbf{A}(x) &= \mathbf{P}^T \mathbf{W}(x) \mathbf{P} = \mathbf{P}^T \mathbf{T}_x^T \mathbf{T}_x \mathbf{W}(x) \mathbf{T}_x^T \mathbf{T}_x \mathbf{P} \\ &= (\mathbf{T}_x \mathbf{P})^T \mathbf{T}_x \mathbf{W}(x) \mathbf{T}_x^T (\mathbf{T}_x \mathbf{P}) \end{aligned} \tag{7}$$

Let's dispart the matrix $(\mathbf{T}_x \mathbf{P})_{n \times m}$ into two parts, as follows

$$\mathbf{T}_x \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix} \tag{8}$$

where matrix $\mathbf{P}_2$ has the size of $(n-k) \times m$ and matrix $\mathbf{P}_1$ has the form

$$\mathbf{P}_1 = \begin{bmatrix} p_1(x_{i1}) p_2(x_{i1}) \ldots p_m(x_{i1}) \\ p_1(x_{i2}) p_2(x_{i2}) \cdots p_m(x_{i2}) \\ \ldots\ldots\ddots\ldots \\ p_1(x_{ik}) p_2(x_{ik}) \cdots p_m(x_{ik}) \end{bmatrix}_{k \times m} \tag{9}$$

Substitute formulas (6) and (8) into (7), and we obtain that coefficient matrix

$$\begin{aligned} \mathbf{A}(x) &= \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}^T \begin{bmatrix} \mathbf{W}_{11}(x) & \mathbf{O}_{12} \\ \mathbf{O}_{21} & \mathbf{O}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix} \\ &= \mathbf{P}_1^T \mathbf{W}_{11}(x) \mathbf{P}_1 . \end{aligned} \tag{10}$$

Due to diagonal matrix $\mathbf{W}_{11}(x)$ being positive symmetry, $\det(\mathbf{A}(x)) \neq 0$ if and only if $rank(\mathbf{P}_1) = m$. Using the Remark 2 and Definition 1, that is to say functions series $\{p_i(x)\}_{i=1}^m$ should be linear independent on the influence domain $P_x = \{x_{i1}, x_{i2}, \cdots, x_{ik}\}$. Conclude this into the following theorem.

Theorem 3 For any $x \in \Omega$, the linear equation system derived from moving least squares approximation exist unique solution if and only if the base functions $\{p_i(x)\}_{i=1}^m$ is linear independent on the influence domain P_x of point x based on the given particles set P_Ω.

Remark 4 The expression (10) of coefficient matrix $\mathbf{A}(x)$ show us that the coefficient matrix of MLS method is positive symmetry under the condition of Theorem 3. And the characteristic leads that much more methods can be used to solve equations system (3).

Remark 5 The theorem is correct for any kind of weight function, random distributed particles, and any dimension of any shape of domain Ω used in the MLS approximation.

Remark 6 To ensure basis $\{p_i(x)\}_{i=1}^m$ of linear space S being linear independence on the influence domain P_x of any point $x \in \overline{\Omega}$, all of the radius $r_j \quad (1 \leq j \leq n)$of the compact support of weight function $W(x,x_j)$ must large enough such that P_x can contain number of m particles at least. That is to say for any point $x \in \overline{\Omega}$ there are number of $k \geq m = \dim(S)$ supports of weight functions which cover the point x, and among of the k centers of the supports, there are m center at least which do not locate on any curve defined by a function in space S.

Although satisfying the condition of the Theorem 3 ensures that the MLS method has unique analysis solution, it does not tell us what the best radius of the compact support of weight function should be to obtain a good approximation. This problem will be discussed in the following several sections.

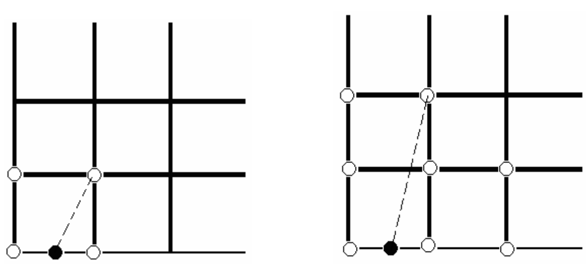
3 Application to linear and quadric basis

Now we use the previous theory to the case of the linear space S with linear and quadric basis respectively and assuming that the particles $P_\Omega = \{x_j\}_{j=1}^n$ are distributed uniformly on the bounded convex domain $\Omega \subset R^2$ with particles step h, namely the distance along the coordinate axis between the adjacent particles.

In the case of linear basis, we have linear space $S = span\{1,x,y\}$ and $m = \dim(S) = 3$. According to Theorem 3 and Remark 6, the influence domain P_x for any point $x \in \Omega$ must include at least 3 particles not sharing any same straight line, then the MLS method can be used to evaluate the approximation of the unknown function at point x. As the particles is distributed like Fig. 1, and weight functions take the form of $w(x,x_j) = w_0\left(\|x-x_j\|_2/r\right)$, the support radius r of the support should be greater than $\sqrt{5}\big/2h \approx 1.11803h$, which can be obtain by sampling several special point x along the domain boundary $\partial\Omega$ only because the interior point x has more particles included in its influence domain .

In the case of quadratic basis, linear space $S = span\{1,x,y,x^2,xy,y^2\}$ and the dimension $m = \dim(S) = 6$. Similarly discussion as in the linear case, we know that the influence domain P_x must include 6 particles not sharing any quadratic curve (two lines regards as a special quadratic curve), and further the support radius r should be greater than $\sqrt{17}/2h \approx 2.0616h$.

Remark 7 Although the previous conditions about the radius of support can ensure the matrix inversion of the MLS approximation possible, a little larger support radius than the previous mentioned is needed to reduce the condition number of the coefficient matrix. Through computing simulation, reference [Zuo and Nie (2005)] suggests that $r \geq 1.2h > \sqrt{5}\big/2h$ for linear basis and $r \geq 2.5h > \sqrt{17}/2h$ for quadratic basis. We will search the best support radius based on a model in the coming section.



(a) linear basis (b) quadratic basis

Figure 1 : The positions where the minimum radius of influence domain needed by MLS method. • point $x \in \overline{\Omega}$ ○ particle in P_x

4 Model of optimal radius

It is very hard just depending on the mathematical analysis to obtain the optimal radius of support of the weight function used in the MLS approximation. To our knowledge there is no a clear result about this problem up to now. Here we try to develop a mathematic model and systemic numerical tests based on approximation theory to solve this problem partially.

According to reference [Atluri (2004)], the 4^{th} order spline function which takes as the following form

$$w_0(t) = \begin{cases} 1-6t^2+8t^3-3t^4 & 0 \leq t \leq 1 \\ 0 & 1 < t \end{cases} \tag{11}$$

is suggested to use as the weight function in MLS method because it has better smoothness of the first derivative of the approximate function.

Owing to Weierstrass theorem that any continuous function can be approximated by a polynomial for any given accuracy requirement, we take the monomial basis, for example in R^2 space they are

$$1,x,y,x^2,xy,y^2,x^3,x^2y,xy^2,y^3,\cdots$$

to express the polynomial which is used to approximate a given functions. Thus a good approximation method should approximate monomials efficiently, and the reverse is correct because of the following fact

$$|f-s^*| \leq |f-p|+|p-s^*| \leq \varepsilon+|p-s^*|,$$

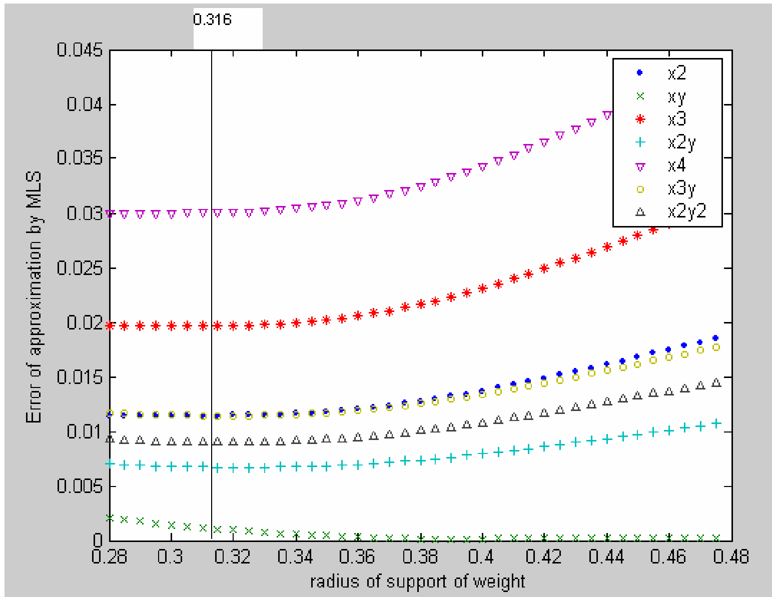
where f is any continuous function, s^* is the approximation function of function f by MLS methods, and p is the polynomial used to approximate f for a given accuracy requirement $\varepsilon > 0$.

For each monomial, we use MLS method to approximate it, and find the best radius through comparing the Sobolev norms of the approximate error. Considering the reproducing ability for polynomials of MLS which is decided by the linear space S used in formula (1), the first several monomials no need to be tested.

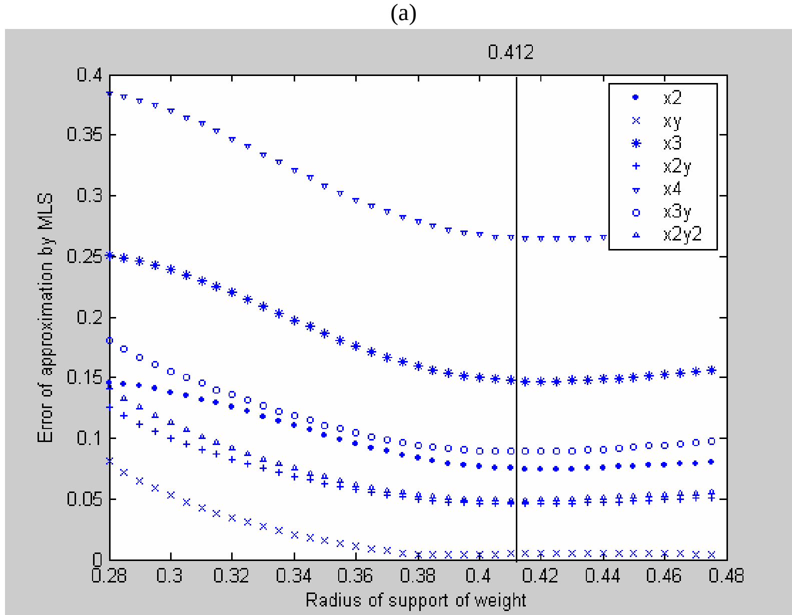
Sobolev norm $\|\bullet\|_t$ is defined as follows

$$\|f\|_t = \left(\sum_{l=0}^{t}\sum_{|\alpha|=l}\int_\Omega [D^\alpha f(\mathbf{x})]^2 d\Omega\right)^{1/2}, \quad f \in H^t(\Omega), \quad \Omega \subset R^d \tag{12}$$

where multi-index $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_d)$ is used, $|\alpha| = \sum_{i=1}^{d}\alpha_i$, $\{\alpha_i\}_{i=1}^{d}$ are nonnegative integers, and differential operator $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$.



(a)



(b)

Figure 2 : Errors of MLS method to monomials with varying support radius (Linear basis). (a) $\|\bullet\|_0$, (b) $\|\bullet\|_1$ norms are used to measure errors respectively.

Denote the **monomial optimal radius** (MOR) of support of the weight used in MLS method as $r_{opt}^{(t,\alpha)}$ while the approximated monomial is $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ and Sobolev norm $\|\bullet\|_t$ is used to measure approximate error. MOR $r_{opt}^{(t,\alpha)}$ can be gotten approximately through simple searching methods and the initial searching radius can be defined through the theory in section 2 (**Step 0**).

Generally, for different monomials, the MORs are different each other. So we need use these MORs to develop the optimal radius of MLS method $r_{opt}^{(t)}$. Assume that the MLS approximate has degree of (m^*-1) polynomial reproducing. The following three steps can be used to obtain $r_{opt}^{(t)}$.

Step 1 Develop the **group optimal radius** (GOR) of

Table 1 : Evaluation of optimal radius in linear basis case (h=0.25)

h=0.25		n=2 x^2	n=2 xy	n=3 x^3	n=3 x^2y	n=4 x^4	n=4 x^3y	n=4 x^2y^2	…
t=0	$r_{opt}^{(0,\alpha)}$	0.290	0.390	0.290	0.325	0.290	0.315	0.315	…
	$r_{opt}^{(0,n)}$	0.323		0.308		0.305			…
	$r_{opt,n}^{(0)}$	0.323		0.318		0.316			…
t=1	$r_{opt}^{(1,\alpha)}$	0.420	0.390	0.420	0.415	0.420	0.415	0.410	…
	$r_{opt}^{(1,n)}$	0.410		0.418		0.416			…
	$r_{opt,n}^{(1)}$	0.410		0.413		0.412			…

Table 2 : Evaluation of optimal radius in quadratic basis case (h=0.25)

h=0.25		n=3 x^3	n=3 x^2y	n=4 x^4	n=4 x^3y	n=4 x^2y^2	n=5 x^5	n=5 x^4y	n=5 x^3y^2	…
t=0	$r_{opt}^{(0,\alpha)}$	0.560	0.520	0.560	0.545	0.515	0.560	0.545	0.515	…
	$r_{opt}^{(0,n)}$	0.540		0.545			0.540			…
	$r_{opt,n}^{(0)}$	0.540		0.542			0.541			…
t=1	$r_{opt}^{(1,\alpha)}$	0.585	0.575	0.590	0.580	0.525	0.630	0.580	0.555	…
	$r_{opt}^{(1,n)}$	0.580		0.573			0.588			…
	$r_{opt,n}^{(1)}$	0.580		0.578			0.579			…

MLS method $r_{opt}^{(t,n)}$ as simple arithmetic average of the MORs of monomials with the same degree n:

$$r_{opt}^{(t,n)} = \sum_{|\alpha|=n} r_{opt}^{(t,\alpha)} \Big/ C_n, \tag{13}$$

where C_n is the cardinality of multi-index set $\{\alpha : |\alpha| = n\}$.

Step 2 Develop the **partial optimal radius** (POR) of MLS method $r_{opt,n}^{(t)}$ as

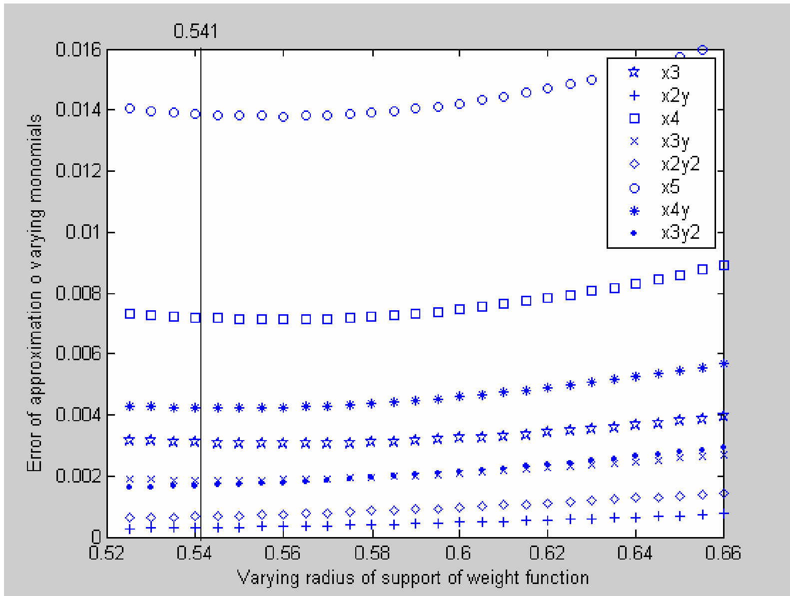
$$r_{opt,n}^{(t)} = \sum_{l=m^*}^{n} w_l r_{opt}^{(t,l)} \Big/ \sum_{l=m^*}^{n} w_l. \tag{14}$$

where $\{w_l\}_{l=m^*}^{n}$ are a group of weights of GOR $\left\{r_{opt}^{(t,l)}\right\}_{l=m^*}^{n}$ For the first several degrees of monomials less than m^*, any radius which satisfies the condition of Theorem 3 in this paper is optimal because of the zero error caused by the reproduction. So the optimal radius formula (14) does not include the contributions of those low orders of monomials. As the equal weights $\{w_l\}_{l=m^*}^{n}$ are used for the GOR, formula (14) means the simple arithmetic average over the nonzero error groups. Otherwise, it is an average with weights on GOR. In order to ensure better approximate ability on a linear space, gradual reducing weights such as $\{w_l = 1/2^l\}_{l=m^*}^{n}$ are suggested to use.

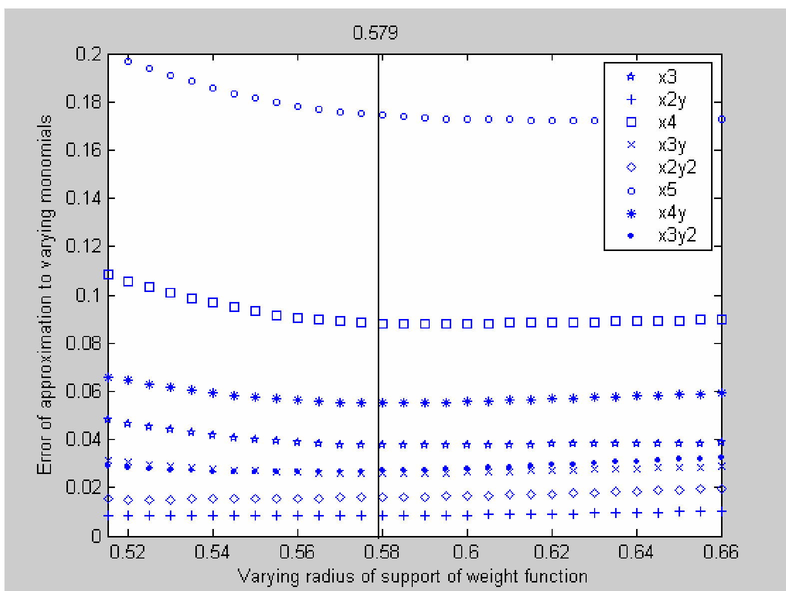
Step 3 Develop the optimal radius of MLS method $r_{opt}^{(t)}$ as the limit of POR

$$r_{opt}^{(t)} = \lim_{n\to\infty} r_{opt,n}^{(t)}. \tag{15}$$

As uniform particles are used, $r_{opt}^{(t)}$ depends on particle



(a)



(b)

Figure 3 : Errors of MLS method to monomials with varying support radius (Quadric basis). (a) $\|\bullet\|_0$, (b) $\|\bullet\|_1$ norms are used to measure errors respectively.

step *h*obviously. Let's name quantity $r_{opt}^{(t)}/h = s^{(t)}$ as optimal scaling. The numerical results in section 6 will show that optimal scaling is independent on particle step *h*.

5 Optimal radius of linear and quadratic basis

Now we use the model posed in the previous section to develop the optimal scaling of MLS in the case of linear space S with linear and quadratic basis being used for 2 dimensional domain $\Omega = [0,\ 1] \times [0,\ 1]$ with uniform distributed particles.

While the linear basis are used in MLS approximation, the errors of approximations to a group of monomials with varying radius in different norms are show in Fig.2, and the MORs $r_{opt}^{(0,\alpha)}$ and $r_{opt}^{(1,\alpha)}$ is given in Tab. 1. In the computing process, searching method is used with researching step 0.005 and initial radius value $1.12h > \sqrt{5}\big/2h \approx 1.11803h$ that the reason has been discussed in details in section 3.

Considering the symmetry, we just show data about the partial monomials in a group in Fig. 2 and Tab. 1. GORs and PORs are evaluated due to formula (13) and (14) with weights $\{w_l = 1/2^l\}_{l=m^*}^{n}$ respectively. Owing to formula (15), the optimal radius of MLS $r_{opt}^{(0)} \approx 0.316, \quad r_{opt}^{(1)} \approx 0.412$ while particle step $h = 0.25$ is used, and the optimal scalings $s^{(0)} \approx 1.264, \quad s^{(1)} \approx 1.648$. The reason that scaling $s^{(1)} > s^{(0)}$ is that error measured by norm $\|\bullet\|_1$ includes more items compared with norm $\|\bullet\|_0$, namely first order differentials, and this leads to more smooth requirement.

While the quadric bases are used, the same process as the linear case is followed except that for the initial searching radius is $2.06h \approx \sqrt{17}\big/2h$. The corresponding data are shown in Fig. 3 and Tab. 2. The optimal radius of MLS is $r_{opt}^{(0)} \approx 0.541, \quad r_{opt}^{(1)} \approx 0.579$ while particle step $h = 0.25$ is used, and the optimal scalings $s^{(0)} \approx 2.164, \quad s^{(1)} \approx 2.316$.

Remark 8 Although the data we used is dependent on node step h, the optimal scaling $s^{(t)}$ are independent of h which will be shown through numerical examples in next section. And the results can be applied to the other 2 dimensional domain with uniform nodes because linear map does not change the key characteristics of polynomials such as degree.

Remark 9 The model can be used to three dimensional problem without any difficulty. As the quasi-uniform [Babuska , Banerjee and Osborn (2003)] distributed particles are used, it can be used with little modify which will be given in coming paper.

6 Numerical test

In this section, we check that the optimal scaling $s^{(t)}$ has little dependent on node step h firstly. Then some complex functions are used to test the efficiency of the optimal scaling obtained by the model posed in this paper.

6.1 *Optimal scaling independent on node step test*

Two different node steps 0.2 and 0.125 compared with the step 0.25 are used to the model in section 5 to obtain the optimal scaling while linear and quadratic base are used respectively. The optimal scaling results evaluated from the data in Tab.5-6 and Tab. 7-8 are given in Tab. 3 and Tab. 4 for two kinds of linear space S respectively. The results show the optimal scaling of radius of support of weight function evaluated from different steps is equal each other approximately, and the little difference among them is caused mainly from searching step.

Table 3 : Comparing of optimal scaling with varying particle steps (linear basis)

h	0.25	0.2	0.125	...
$s^{(0)}$	1.264	1.275	1.264	...
$s^{(1)}$	1.648	1.665	1.672	...

Table 4 : Comparing of optimal scaling with varying particle steps (quadratic basis)

h	0.25	0.2	0.125	...
$s^{(0)}$	2.164	2.175	2.176	...
$s^{(1)}$	2.316	2.315	2.200	...

6.2 *Efficient test*

Three functions $e^x \sin y \sin(x^2 y)$ and e^{xy^3} are used to test the efficiency of the optimal scaling of support radius of weight function in MLS method. And let particle step h=0.125. The error curves measured by two kinds of norms varying with support radius are displayed in Fig. 4 for linear basis case and Fig. 5 for quadratic basis.

Fig. 4 (a) and (b) show that while support radius

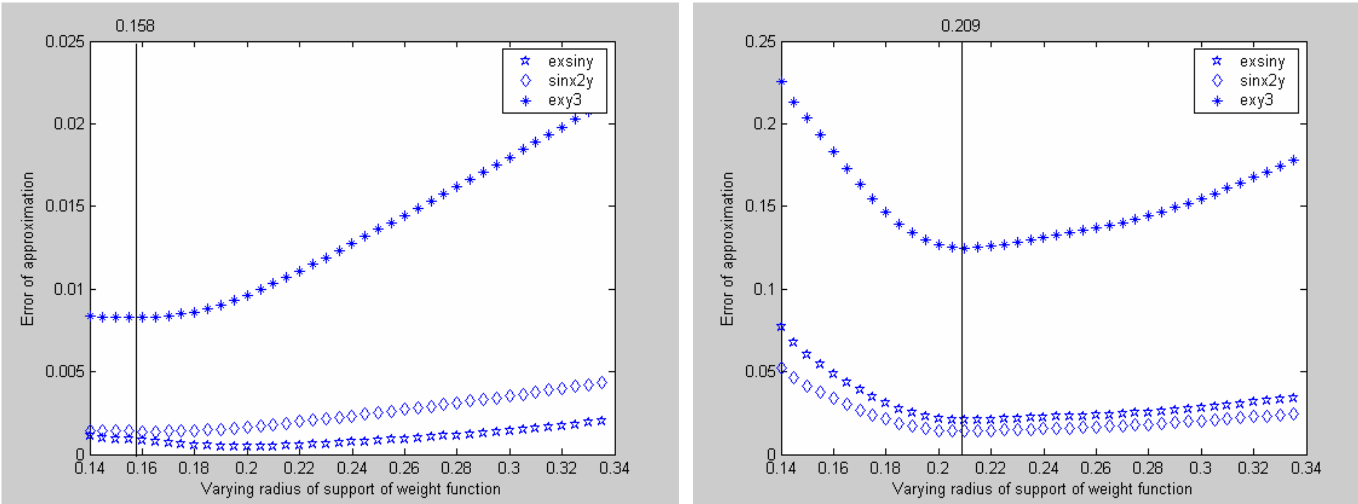
$$r = s^{(0)}h = 1.264 \times 0.125 = 0.158$$

and

$$r = s^{(1)}h = 1.672 \times 0.125 = 0.209$$

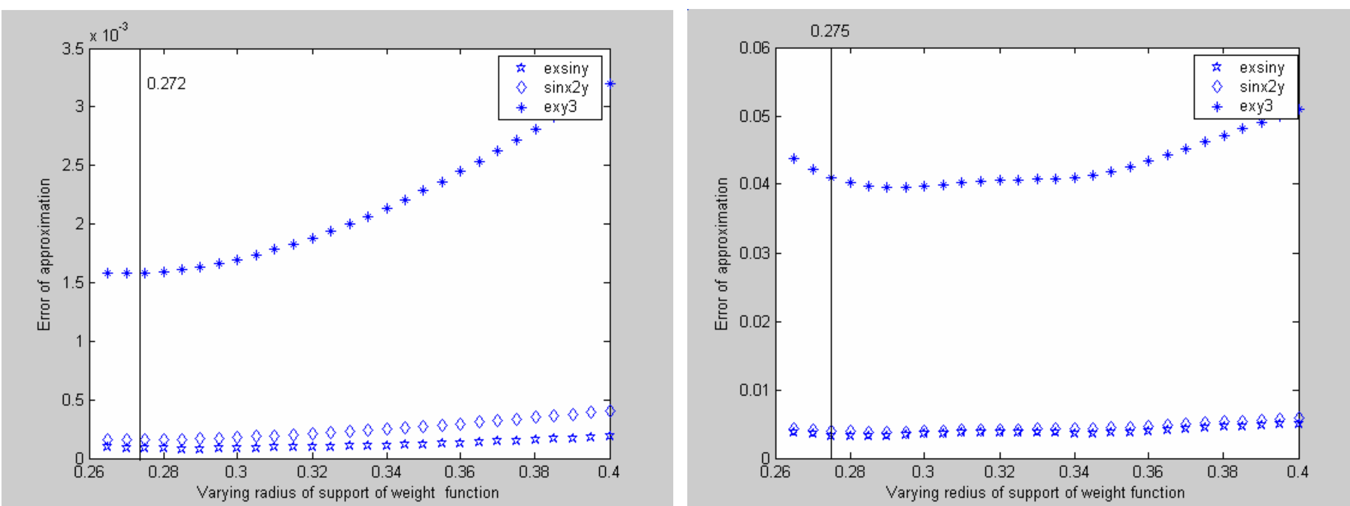
are used respectively MLS method with linear basis gives the three test functions the best approximations all-around in corresponding norm of $\|\bullet\|_0$ and $\|\bullet\|_1$. Similarly, Fig. 5 (a) and (b) indicate that while MLS method with quadratic basis and support radius taken as

$$r = s^{(0)}h = 2.176 \times 0.125 = 0.272$$



(a) (b)

Figure 4 : Test about optimal radius (linear basis case). (a) $\|\bullet\|_0$, (b) $\|\bullet\|_1$ norms are used to measure errors respectively.



(a) (b)

Figure 5 : Test about optimal radius (quadratic basis case). (a) $\|\bullet\|_0$, (b) $\|\bullet\|_1$ norms are used to measure errors respectively.

and

$$r = s^{(1)}h = 2.200 \times 0.125 = 0.275 \quad ,$$

the best approximations all-around to the three test functions in norm of $\|\bullet\|_0$ and $\|\bullet\|_1$ are obtained respectively.

7 Conclusion

Based on matrix analysis and approximation theory, this paper develops an efficient approach to find the optimal radius of support of radial weight function used in moving least squares method. As an example, while uniform distributed particles and the 4^{th} order spline weight function are adopted in MLS method in two dimension domain, and two kinds of norms are used to measure error, optimal results for linear and quadratic basis are obtained, and then the test data verify that the optimal value are correct.

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Table 5 : Evaluation of optimal radius in linear basis case

h=0.2		n=2 x^2	n=2 xy	n=3 x^3	n=3 x^2y	n=4 x^4	n=4 x^3y	n=4 x^2y^2	…
t=0	$r_{opt}^{(0,\alpha)}$	0.235	0.310	0.235	0.260	0.235	0.255	0.255	…
	$r_{opt}^{(0,n)}$	0.260		0.248		0.247			…
	$r_{opt,n}^{(0)}$	0.260		0.256		0.255			…
t=1	$r_{opt}^{(1,\alpha)}$	0.340	0.310	0.340	0.335	0.340	0.335	0.330	…
	$r_{opt}^{(1,n)}$	0.330		0.338		0.336			…
	$r_{opt,n}^{(1)}$	0.330		0.333		0.333			…

Table 6 : Evaluation of optimal radius in linear basis case

h=0.125		n=2 x^2	n=2 xy	n=3 x^3	n=3 x^2y	n=4 x^4	n=4 x^3y	n=4 x^2y^2	…
t=0	$r_{opt}^{(0,\alpha)}$	0.145	0.195	0.145	0.160	0.145	0.160	0.160	…
	$r_{opt}^{(0,n)}$	0.162		0.153		0.154			…
	$r_{opt,n}^{(0)}$	0.162		0.159		0.158			…
t=1	$r_{opt}^{(1,\alpha)}$	0.215	0.195	0.210	0.210	0.210	0.210	0.210	…
	$r_{opt}^{(1,n)}$	0.208		0.210		0.210			…
	$r_{opt,n}^{(1)}$	0.208		0.209		0.209			…

Table 7 : Evaluation of optimal radius in quadratic basis case

h=0.2		n=3 x^3	n=3 x^2y	n=4 x^4	n=4 x^3y	n=4 x^2y^2	n=5 x^5	n=5 x^4y	n=5 x^3y^2	…
t=0	$r_{opt}^{(0,\alpha)}$	0.450	0.415	0.450	0.440	0.415	0.450	0.440	0.415	…
	$r_{opt}^{(0,n)}$	0.433		0.439			0.435			…
	$r_{opt,n}^{(0)}$	0.433		0.435			0.435			…
t=1	$r_{opt}^{(1,\alpha)}$	0.460	0.485	0.460	0.460	0.415	0.475	0.460	0.455	…
	$r_{opt}^{(1,n)}$	0.473		0.451			0.463			…
	$r_{opt,n}^{(1)}$	0.470		0.464			0.463			…

Table 8 : Evaluation of optimal radius in quadratic basis case

h=0.125		n=3 x^3	n=3 x^2y	n=4 x^4	n=4 x^3y	n=4 x^2y^2	n=5 x^5	n=5 x^4y	n=5 x^3y^2	…
t=0	$r_k^{(0,n)}$	0.280	0.260	0.280	0.280	0.255	0.280	0.280	0.265	…
	$r_{opt}^{(0,n)}$	0.270		0.275			0.275			…
	$r_{opt,n}^{(0)}$	0.270		0.272			0.272			…
t=1	$r_k^{(1,n)}$	0.285	0.255	0.285	0.285	0.255	0.290	0.285	0.285	…
	$r_{opt}^{(1,n)}$	0.270		0.279			0.287			…
	$r_{opt,n}^{(1)}$	0.270		0.273			0.275			…

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